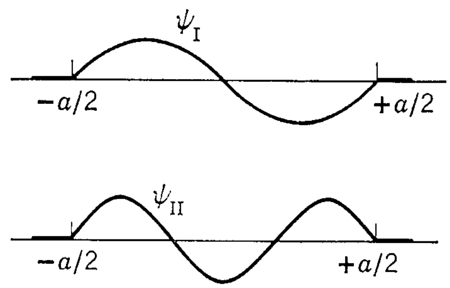
# PHYS 234: Quantum Physics 1 (Fall 2008)
# Assignment 9 – Solutions

Issued: November 14, 2008 Due: 12.00pm, November 21, 2008

1. Two possible eigenfunctions for a particle moving freely in a region of length $a$, but strictly confined to that region, are shown in the figure below. When the particle is in the state corresponding to the eigenfunction $\psi_{\mathrm{I}}$, its total energy is 4 eV.




a) What is its total energy in the state corresponding to $\psi_{\mathrm{II}}$ ?

b) What is the lowest possible energy for the particle in this system ?

**solution** a) In the lowest energy state $n = 1$, $\psi$ has no nodes. Hence $\psi_I$ must correspond to $n = 2$, and $\psi_{II}$ to $n = 3$. Since the energy of the $n^{th}$ state $E_n \propto n^2$ and $E_I = 4$ eV, then

$$\frac{E_{II}}{E_I} = \frac{3^2}{2^2}; \qquad E_{II} = 9\text{eV}$$

b) By the same analysis,

$$\frac{E_0}{E_I} = \frac{1^2}{2^2}; \qquad E_{II} = 1\text{eV}$$

2. It can be proved that, in general, the *inner product* of any two different eigenfunctions (of the same given potential) is always zero. This property is called *orthogonality*. The inner product is defined as,

$$\int_{-\infty}^{\infty} \psi_m(x)^* \psi_n(x) dx = 0 \qquad\qquad m \neq n.$$

Form the inner product of the eigenfunctions for the $n = 1$ and $n = 3$ states of the infinite square well potential. Show that the inner product is equal to zero. In other words, show that

$$\int_{-\infty}^{\infty} \psi_1^*(x)\psi_3(x) dx = 0$$

Hint: The following identities may be useful:

$$\cos(v)\cos(v) = [\cos(u+v) + \cos(u-v)]/2)$$

$$\cos(u) - \cos(v) = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

**solution** The inner product that is formed is analysed as follows

$$\int_{-\infty}^{+\infty} \psi_1^* \psi_3 dx = \frac{2}{a}\int_{-a/2}^{+a/2} \cos(\frac{\pi x}{a})\cos(\frac{3\pi x}{a}) dx$$

The integral is simplified using the substitution $u = \frac{\pi x}{a}$ and $v = \frac{3\pi x}{a}$ and the relationship from the problem sheet $\cos(v)\cos(v) = [\cos(u+v) + \cos(u-v)]/2$, hence

$$\int_{-\infty}^{+\infty} \psi_1^* \psi_3 dx = \frac{1}{a}\int_{-a/2}^{+a/2} \cos(\frac{4\pi x}{a}) + \cos(\frac{2\pi x}{a}) dx$$

Both terms in the integral integrate to zero over the range of integration, hence

$$\int_{-\infty}^{+\infty} \psi_1^* \psi_3 dx = 0$$

which is the orthogonality property which we needed to prove.

3. The wave function for a particle is

$$\Psi(x,t) = \sin(kx)[i\cos(\omega t/2) + \sin(\omega t/2)]$$

where $k$ and $\omega$ are constants.

a) Is this particle in a state of definite momentum ? If so, determine the momentum.

b) Is this particle in a state of definite energy ? If so, determine the energy.

**solution** A wave function representing a particle with a definite value for an observable is an eigenfunction of the operator for that observable, with the eigenvalue being the value of that observable and hence the result of a measurement.

a) Apply the momentum operator:

$$-i\hbar\frac{\partial}{\partial x}\sin(kx)[i\cos(\omega t/2) + \sin(\omega t/2)] = i\hbar k\cos(kx)[i\cos(\omega t/2) + \sin(\omega t/2)]$$

This is not a constant times the original wave function, and so not an eigenfunction of the momentum operator and therefore the particle is not in a state with well-defined momentum.

b) Apply the energy operator:

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\sin(kx)[i\cos(\omega t/2) + \sin(\omega t/2)] &= i\hbar\sin(kx)[-i(\omega/2)\sin(\omega t/2) + (\omega/2)\cos(\omega t/2)] \\ &= (\hbar\omega/2)\sin(kx)[i\cos(\omega t/2) + \sin(\omega t/2)] \end{aligned}$$

This is a constant times the original wave function, and so the wave function is an eigenfunction of the energy operator, and therefore the particle is in a state of definite energy. The value of the energy is the eigenvalue, $E = \hbar\omega/2$

4. Using the first two normalized wave functions $\Psi_1(x,t)$ and $\Psi_2(x,t)$ for a particle moving freely in a region of length $a$, but strictly confined to that region, construct the linear combination

$$\Psi(x,t) = c_1\Psi_1(x,t) + c_2\Psi_2(x,t)$$

which is a superposition of the first two energy eigenstates. Then derive a relation involving the adjustable constants $c_1$ and $c_2$ which, when satisfied, will ensure that $\Psi(x,t)$ is also normalised.

**solution** The wavefunctions in question are

$$\Psi_1 = \sqrt{\frac{2}{a}}\cos(\pi x/a)\exp(-iE_1t/\hbar); \Psi_2 = \sqrt{\frac{2}{a}}\sin(2\pi x/a)\exp(-iE_2t/\hbar)$$

with $E_2 = 4E_2$. The linear combination is

$$\Psi = c_1\Psi_1 + c_2\Psi_2$$

Normalising this gives

$$1 = \int_{-\infty}^{\infty}\Psi^*\Psi dx$$

Substituting and expanding, this becomes

$$c_1c_1^*\int_{-\infty}^{\infty}\Psi_1^*\Psi_1 dx + c_2c_2^*\int_{-\infty}^{\infty}\Psi_2^*\Psi_2 dx + c_1^*c_2\int_{-\infty}^{\infty}\Psi_1^*\Psi_2 dx + c_2^*c_1\int_{-\infty}^{\infty}\Psi_2^*\Psi_1 dx = 1$$

Since $\Psi_1$ and $\Psi_2$ are already normalised,

$$\int_{-\infty}^{\infty}\Psi_1^*\Psi_1 dx = \int_{-\infty}^{\infty}\Psi_2^*\Psi_2 dx = 1$$

And the real parts of $\int_{-\infty}^{\infty}\Psi_1^*\Psi_2 dx$ and $\int_{-\infty}^{\infty}\Psi_2^*\Psi_1 dx$ are

$$\int_{-a/2}^{+a/2}\cos(\pi x/a)\sin(2\pi x/a)dx = \frac{a}{\pi}\int_{-\pi/2}^{+\pi/2}\cos(u)\sin(2u)du = 0$$

Then, in order for $\Psi$ to be normalised, it is necessary that

$$c_1c_1^* + c_2c_2^* = 1$$

5. A particle in the infinite square well has as its initial wave function an even mixture of the first two stationary states:

$$\Psi(x,0) = A[\psi_1(x) + \psi_2(x)]$$

(a) Normalise $\Psi(x,0)$. That is, find $A$. (don't forget that $\psi(x)$'s are orthonormal)

**solution**

$$|\Psi|^2 = \Psi^*\Psi = |A|^2(\psi_1^* + \psi_2^*)(\psi_1 + \psi_2) = |A|^2(\psi_1^*\psi_1 + \psi_1^*\psi_2 + \psi_2^*\psi_1 + \psi_2^*\psi_2)$$

$$1 = \int |\Psi|^2 dx = |A|^2 \int (\psi_1^*\psi_1 + \psi_1^*\psi_2 + \psi_2^*\psi_1 + \psi_2^*\psi_2) dx = 2|A|^2$$

Hence,

$$A = \frac{1}{\sqrt{2}}$$

(b) Find $\Psi(x,t)$ and $|\Psi(x,t)|^2$. Express the latter as a sinusoidal function of time using the simplification that $\omega = \pi^2\hbar/2ma^2$

**solution**

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left[\psi_1 \exp\{-iE_1t/\hbar\} + \psi_2 \exp\{-iE_2t/\hbar\}\right]$$

Using $\psi_n = \sin(n\pi x/a)$ and $E_n/\hbar = n^2\omega$, this simplifies to

$$\frac{1}{\sqrt{a}} \exp\{-i\omega t\} \left[\sin\left(\frac{\pi}{a}x\right) + \sin\left(\frac{2\pi}{a}x\right) \exp\{-3i\omega t\}\right]$$

$$\begin{aligned} |\Psi(x,t)|^2 &= \frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)(\exp\{-3i\omega t\} + \exp\{3i\omega t\}) + \sin^2\left(\frac{2\pi}{a}x\right)\right] \\ &= \frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin^2\left(\frac{2\pi}{a}x\right) + 2\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos(3\omega t)\right] \end{aligned}$$

(c) Compute $< x >$. Notice that it oscillates in time. What is the angular frequency of the oscillation? What is the amplitude of the oscillation?

**solution**

$$\begin{aligned} < x > &= \int x|\Psi(x,t)|^2 dx \\ &= \frac{1}{a}\int_0^a x\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin^2\left(\frac{2\pi}{a}x\right) + 2\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos(3\omega t)\right] dx \end{aligned}$$

First term integrate by parts

$$\int_0^a x\sin^2\left(\frac{\pi}{a}x\right) dx = \left[\frac{x^2}{4} - \frac{x\sin(2\pi x/a)}{4\pi/a} - \frac{\cos(2\pi x/a)}{8(\pi/a)^2}\right]_0^a = \frac{a^2}{4}$$

Second term amounts to the same value

$$\int_0^a x \sin^2\left(\frac{2\pi}{a}x\right) dx = \frac{a^2}{4}$$

Third term requires an identity to make the integral tractable

$$\int_0^a \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{2\pi}{a}x\right) dx = \frac{1}{2}\int_0^a x\left[\cos(\pi x/a) - \cos(3\pi x/a)\right] dx$$
$$= \frac{1}{2}\left[\frac{a^2}{\pi^2}\cos(\pi x/a) + \frac{ax}{\pi}\sin(\pi x/a) - \frac{a^2}{9\pi^2}\cos(3\pi x/a) - \frac{ax}{3\pi}\sin(3\pi x/a)\right]_0^a$$
$$= \frac{1}{2}\left[\frac{a^2}{\pi^2}(\cos(\pi) - \cos(0) - \frac{a^2}{9\pi^2}(\cos(3\pi x/a) - \cos(0))\right] = -\frac{8a^2}{9\pi^2}$$

Hence the expectation value is

$$< x >= \frac{1}{a}\left[\frac{a^2}{4} + \frac{a^2}{4} - \frac{16a^2}{9\pi^2}\cos(3\omega t)\right] = \frac{a}{2}\left[1 - \frac{32}{9\pi^2}\cos(3\omega t)\right]$$

This is a function of time, with

$$\text{Amplitude } = \frac{32}{9\pi^2}\frac{a}{2}$$

$$\text{angular frequency } = 3\omega = \frac{3\pi^2\hbar}{2ma^2}$$

(d) If you measured the energy of this particle, what values might you get, and what is the probability of getting each of them? Find the expectation value of $H$. How does it compare with $E_1$ and $E_2$?

**solution** You could get either $E_1 = \pi^2\hbar^2/2ma^2$ or $E_2 = 2\pi^2\hbar^2/ma^2$ with equal probability $P_1 = P_2 = 1/2$.
So,

$$< H >= \frac{1}{2}(E_1 + E_2) = \frac{5\pi^2\hbar^2}{4ma^2}$$

Which is the average of $E_1$ and $E_2$