PHYS 234: Quantum Physics 1 (Fall 2008)
Assignment 8

Issued: November 7, 2008 Due: 12.00pm, November 14, 2008

1. **A time dependent two level system**
An atom that can exist in one of two energy states (or levels) can be described similar to the spin of a silver atom. The two orthonormal basis states for the atom are the ground state (lowest energy), $\big|\,g\big\rangle$, and the excited state $\big|\,e\big\rangle$. In the following we consider the interaction between a two-level atom and a light field which can change the energy of the atom and move it between the two states. In the so-called *r*otating wave approximation, the Hamiltonian of the two-level system is given by

$$H = \hbar\omega_g\big|g\big\rangle\big\langle g\big| + \hbar\omega_e\big|e\big\rangle\big\langle e\big| - \hbar\Omega_R(\exp\{-i\nu t\}\big|g\big\rangle\big\langle e\big| + \exp\{i\nu t\}\big|e\big\rangle\big\langle g\big|)$$

The first two terms describe the dynamics of the two-level atom in the absence of the light field, whereas the interaction between the two is encoded in the last two terms. In the following we parameterise the state of the two-level system by

$$\big|\,\psi(t)\big\rangle = C_g(t)\big|\,g\big\rangle + C_e(t)\big|\,e\big\rangle$$

and only consider the case where the frequency of the light field is resonant with the atom:

$$\omega_g - \omega_e = \nu$$

(a) Express the Hamiltonian in co-ordinate representation with respect to the basis states $\big|\,g\big\rangle$ and $\big|\,e\big\rangle$

solution

$$\hat{H} = \begin{pmatrix} \big\langle g\,\big|H\big|\,g\big\rangle & \big\langle g\,\big|H\big|\,e\big\rangle \\ \big\langle e\,\big|H\big|\,g\big\rangle & \big\langle e\,\big|H\big|\,e\big\rangle \end{pmatrix} = \begin{pmatrix} \hbar\omega_g & \hbar\Omega_R\exp\{-i\nu t\} \\ \hbar\Omega_R(\exp\{i\nu t\} & \hbar\omega_e \end{pmatrix}$$

(b) Use the Schroedinger equation (for probability amplitudes) to obtain the coupled differential equations that govern the time dependence of the coefficients $C_g(t)$ and $C_e(t)$.

solution The Schroedinger equation in question reads

$$i\hbar\frac{\partial}{\partial t}\vec{\psi}(t) = \hat{H}\vec{\psi}(t)$$

where $\vec{\psi}(x,t)$ is the column vector of probability amplitudes in this discrete case (in the case of continuous eigenvectors (e.g. position), the column vector becomes a function – the wave function).

The state ket presented in the question leads to the following vector of probability amplitudes in the ordered basis $(\big|\,g\big\rangle, \big|\,e\big\rangle)$

$$\big|\,\psi(t)\big\rangle = C_g(t)\big|\,g\big\rangle + C_e(t)\big|\,e\big\rangle \quad\Rightarrow\quad \vec{\psi}(t) = \begin{pmatrix} C_g(t) \\ C_e(t) \end{pmatrix}$$

This, combined with the Schroedinger equation above, leads to the coupled differential equations

$$\dot{C}_g = -i\omega_g C_g + i\Omega_R\exp\{-i\nu t\}C_e \tag{1}$$
$$\dot{C}_e = -i\omega_e C_e + i\Omega_R\exp\{i\nu t\}C_g \tag{2}$$

(c) To solve these differential equations it is useful to consider the so-called *interaction picture*, where the functions $C_g(t)$ and $C_e(t)$ are given by

$$C_g(t) = D_g(t)\exp\{-i\omega_g t\}, \quad C_e(t) = D_e(t)\exp\{-i\omega_e t\}$$

By considering the time derivative of the above expressions, form a new set of coupled equations for $D_g(t)$ and $D_e(t)$. This formalism is used to eliminate the first two parts of the Hamiltonian which only describe the dynamics of the two-level system. What is left concerns the interaction between the atom and the light field.

solution

$$\begin{aligned}
\dot{C}_g &= -i\omega D_g(t)\exp\{-i\omega_g t\} + \dot{D}_g(t)\exp\{-i\omega_g t\} \\
&= -i\omega_g C_g + \dot{D}_g(t)\exp\{-i\omega_g t\} \\
\text{equate to (1)} \quad \Rightarrow \quad \dot{D}_g(t)\exp\{-i\omega_g t\} &= i\Omega_R \exp\{-i\nu t\} C_e \\
&= i\Omega_R \exp\{-i\nu t\} D_e(t)\exp\{-i\omega_e t\} \\
\dot{D}_g(t) &= i\Omega_R \exp\{-i\nu t\} D_e(t)\exp\{-i(\omega_e + \omega_g)t\} \\
\dot{D}_g(t) &= i\Omega_R D_e(t)
\end{aligned}$$

Similarly

$$\dot{D}_e(t) = i\Omega_R D_g(t)$$

(d) Solve the resulting coupled differential equations and use the initial conditions $D_g(0)$ and $D_e(0)$ to fix any arbitrary constants.

solution The differential equations can be decoupled by defining $D_\pm = D_g \pm D_e$, which leads to the following differential equations

$$\dot{D}_+ = i\Omega_R D_+ \quad \text{and} \quad \dot{D}_- = -i\Omega_R D_-$$

Solving these two equations with initial conditions given in the question gives

$$\begin{aligned}
D_g(t) &= D_g(0)\cos(\Omega_R t) + iD_e(0)\sin(\Omega_R t) \\
D_e(t) &= iD_g(0)\sin(\Omega_R t) + D_e(0)\cos(\Omega_R t)
\end{aligned}$$

(e) Compute the probability to find the atom in the ground state, given by $|\langle g|\psi\rangle|^2$, for the case that the atom was initially in the excited state, $|\psi(0)\rangle = |e\rangle$

solution The initial state has $C_g(0) = 0$ and $C_e(0) = 1$, which converts to $D_g(0) = 0$ and $D_e(0) = 1$. The probability that the atom is in the ground state

$$Prob\left(|g\rangle\right) = |D_g(t)|^2 = \sin^2(\Omega_R t)$$

2. **Wave functions**
A particle of mass m is in the state

$$\Psi(x,t) = A\exp\{-a[(mx^2/\hbar) + it]\}$$

where A and a are positive real constants

(a) Find A

solution A is found by ensuring that the wave function is normalised in the sense that the probability density equals unity when integrated over all x

$$1 = |A|^2 \int_{-\infty}^{+\infty} \exp\{-2amx^2/\hbar\}dx = |A|^2\sqrt{\frac{\pi}{2am/\hbar}} \quad \Rightarrow \quad A = \left(\frac{2am}{\pi\hbar}\right)^{1/4}$$

(b) Since the total energy is the sum of kinetic and potential energies, the Hamiltonian operator is written as

$$H = \frac{P^2}{2m} + V(X) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

For what potential energy function $V(x)$ does Ψ satisfy the Schroedinger equation?

solution

$$\frac{\partial\Psi}{\partial t} = -ia\Psi; \quad \frac{\partial\Psi}{\partial x} = -\frac{2amx}{\hbar}\Psi; \quad \frac{\partial^2\Psi}{\partial x^2} = -\frac{2am}{\hbar}\left(\Psi + x\frac{\partial\Psi}{\partial x}\right) = -\frac{2am}{\hbar}\left(1 - \frac{2amx^2}{\hbar}\right)\Psi$$

Put these into the Schroedinger Equation: $i\hbar\frac{\partial\Psi}{\partial t} = \frac{-hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi$:

$$\begin{aligned} V(x)\Psi &= i\hbar(-ia)\Psi + \frac{-hbar^2}{2m}\left(-\frac{2am}{\hbar}\right)\left(1 - \frac{2amx^2}{\hbar}\right)\Psi \\ &= \left[\hbar a - \hbar a\left(1 - \frac{2amx^2}{\hbar}\right)\right]\Psi = 2a^2mx^2\Psi \quad \Rightarrow \quad V(x) = 2ma^2x^2 \end{aligned}$$

This is the potential associated with the harmonic oscillator, $V(x) \sim x^2$, e.g. for a mass-spring system, Potential Energy $= 1/2kx^2$

(c) Calculate the expectation values for the first and second moments of position and momentum, $< x >, < x^2 >, < p >$ and $< p^2 >$

solution

$$< x >= \int_{-\infty}^{+\infty} dx x|\Psi|^2 = 0$$

Particle is equally likely to be on the $-x$ as the $+x$ side of the potential that is symmetric about $x = 0$

$$< x^2 >= \int_{-\infty}^{+\infty} dx x^2|\Psi|^2 = |A|^2\frac{1}{2(2am/\hbar)}\sqrt{\frac{\pi\hbar}{2am}} = \frac{\hbar}{4am}$$

$$< p >= \int_{-\infty}^{+\infty} dx\Psi^*(-i\hbar)\frac{\partial}{\partial x}\Psi = 0$$

Derivative evaluated in part (b), so integral is same as for $< x >$. Particle is equally likely to be moving in direction of positive x as negative x.

$$\begin{aligned} < p^2 > &= \int_{-\infty}^{+\infty} dx \Psi^* (-i\hbar)^2 \frac{\partial^2}{\partial x^2} \Psi \\ &= 2amh \left[\int_{-\infty}^{+\infty} dx |\Psi|^2 - \frac{2am}{\hbar} \int_{-\infty}^{+\infty} dx x^2 |\Psi|^2 \right] \\ &= 2am\hbar \left(1 - \frac{2am}{\hbar} < x^2 > \right) = am\hbar \end{aligned}$$

(d) Find the variances in position and momentum, Δx and Δp. Is their product consistent with the Heisenberg uncertainty principle?

solution

$$\Delta X = \sqrt{< X^2 > - < X >^2} = \sqrt{\frac{\hbar}{4am}} \qquad \Delta P = \sqrt{< P^2 > - < P >^2} = \sqrt{am\hbar}$$

Consequently the product of these uncertainties is

$$\Delta X \Delta P = \sqrt{\frac{\hbar}{4am}} \sqrt{am\hbar} = \frac{\hbar}{2}$$

Which is the equality limit of the Heisenberg Uncertainty principle. Note that the Wave function is a gaussian wave-packet.

3. **Heisenberg Uncertainty Principle**
In this question you will use the Heisenberg uncertainty principle to estimate certain parameters. The values obtained stem purely from quantum physics and have no classical analogue.

(a) Consider an electron whose position is is somewhere inside an atom of diameter $1 \ \mathring{A}$. What is the uncertainty in the electron's momentum? Is this consistent with the binding energy of electrons in atoms. [To make the comparison, assume the electron has an momentum equal to its minimum uncertainty]

solution Set $\Delta X = 10^{-10}$ m

$$P \sim \Delta P = \frac{\hbar}{2\Delta X}$$

$$E = \frac{P^2}{2m} = \left(\frac{\hbar}{2\Delta X} \right)^2 \frac{1}{2m} = 0.95 \text{ eV}$$

Atomic binding energies are typically on the order of a few electron volts, so this result is consistent with finding electrons inside atoms.

(b) Imagine an electron to be somewhere in a nucleus of diameter 10^{-12} cm. What is the uncertainty in the electron's momentum? Is the resulting energy consistent with binding energy of nuclear constituents and therefore would you expect the electron to escape the nucleus. [The conversion from momentum to energy must be done relativistically in this case: $E_{tot}^2 = c^2 p^2 + m_0^2 c^4$ and $E_{kinetic} = E_{tot} - m_0 c^2$]

solution Set $\Delta X = 10^{-14}$ m

$$E_{tot} = \sqrt{\left(\frac{\hbar c}{2\Delta X}\right)^2 + m_0^2 c^4} = 9.88 \text{ MeV}$$

$$E_{kinetic} = 9.88 - 0.511 = 9.37 \text{ MeV}$$

This value is close to the average binding energy per nucleon, so electrons will tend to escape from the nuclei.

(c) Consider now a neutron or a proton to be in such a nucleus. What is the uncertainty in the neutron or proton's momentum? Is this consistent with the binding energy of nuclear constituents? (Again, a relativistic calculation is required here).

solution The estimated momentum is the same as in part (b) since it depends only on the uncertainty in position (ΔX). However the rest energy (938 MeV) is much greater and dominates the total energy and consequently the kinetic energy portion is much smaller

$$E_{tot} = \sqrt{\left(\frac{\hbar c}{2\Delta X}\right)^2 + m_0^2 c^4} = 938.052 \text{ MeV}$$

$$E_{kinetic} = 938.052 - 938 = 0.052 \text{ MeV}$$

This is much smaller than the average binding energy per nucleon, thus the uncertainty principle is consistent with finding these particles confined inside the nucleus.