Assignment 7 – Solutions.

1(a) Eigenvalues of $\hat{H}$ (note factor of $\hbar\omega$) :

$$\begin{vmatrix} 0-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 0-\lambda \end{vmatrix} = 0$$

$$-\lambda(2-\lambda)(-\lambda) + (-1)(2-\lambda) = 0$$

$$(2-\lambda)(\lambda^2 - 1) = (2-\lambda)(\lambda-1)(\lambda+1) = 0$$

$\Rightarrow \lambda_1 = 2\hbar\omega, \quad \lambda_2 = \hbar\omega, \quad \lambda_3 = -\hbar\omega$

(b) Eigenvectors

$\lambda_1 = 2\hbar\omega$:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \\ 2c \end{pmatrix} \Rightarrow \begin{aligned} c &= 2a \\ 2b &= 2b \\ a &= 2c \end{aligned}$$

$c = a = 0, \quad b = 1 \quad \Rightarrow |\lambda_1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_2 = \hbar\omega$:

$$\left.\begin{aligned} c &= a \\ 2b &= b \\ a &= c \end{aligned}\right\} \quad b = 0, \quad c = a = 1 \quad \Rightarrow |\lambda_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

normalise

$\lambda_3 = -\hbar\omega$:

$$\left.\begin{aligned} c &= -a \\ 2b &= -b \\ a &= -c \end{aligned}\right\} \quad b = 0, \quad c = -a$$

$c = -1, \; a = +1 \qquad |\lambda_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

(b) $\hat{U} = \exp\left\{-\frac{i\hat{H}t}{\hbar}\right\}$

Using eigenvectors & eigenvalues of $\hat{H}$

$$\hat{U} = \exp\{-i2\omega t\}\begin{pmatrix}0\\1\\0\end{pmatrix}(0\ 1\ 0)$$

$$+ \exp\{-i\omega t\}\frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix}(1\ 0\ 1)$$

$$+ \exp\{+i\omega t\}\frac{1}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix}(1\ 0\ -1)$$

$$= \exp\{-i2\omega t\}\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix} + \frac{\exp\{-i\omega t\}}{2}\begin{pmatrix}1&0&1\\0&0&0\\1&0&1\end{pmatrix} + \frac{\exp\{+i\omega t\}}{2}\begin{pmatrix}1&0&-1\\0&0&0\\-1&0&1\end{pmatrix}$$

$$= \begin{pmatrix}\frac{1}{2}\left(\exp\{-i\omega t\}+\exp\{+i\omega t\}\right) & 0 & \frac{1}{2}\left(\exp\{-i\omega t\}-\exp\{+i\omega t\}\right)\\ 0 & \exp\{-i2\omega t\} & 0\\ \frac{1}{2}\left(\exp\{-i\omega t\}-\exp\{+i\omega t\}\right) & 0 & \frac{1}{2}\left(\exp\{-i\omega t\}+\exp\{+i\omega t\}\right)\end{pmatrix}$$

$$= \begin{pmatrix}\cos(\omega t) & 0 & -i\sin(\omega t)\\ 0 & \exp\{-i2\omega t\} & 0\\ -i\sin(\omega t) & 0 & \cos(\omega t)\end{pmatrix}$$

(c) $|\psi(t)\rangle = U|\psi(t_0)\rangle$

$\vec{\psi}(t) = \hat{U}\vec{\psi}(t_0)$ with respect to particular basis.

$$= \begin{pmatrix} \cos(\omega t) & 0 & -i\sin(\omega t) \\ 0 & \exp\{-i2\omega t\} & 0 \\ -i\sin(\omega t) & 0 & \cos(\omega t) \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$= \begin{pmatrix} a\cos(\omega t) - ic\sin(\omega t) \\ b\exp\{-i2\omega t\} \\ c\cos(\omega t) - ia\sin(\omega t) \end{pmatrix}$$

Assignment 7 – SOLUTIONS

2 (a) Schrödinger Equation: $i\hbar \frac{\partial}{\partial t} \vec{\Psi}(t) = \hat{H} \vec{\Psi}(t)$

$$\Rightarrow \quad i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \hbar\omega \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

$$\Rightarrow \quad \dot{\alpha}_1 = -i\omega \alpha_3 \qquad \left(\text{where } \frac{\partial}{\partial t} a = \dot{a} \right)$$

$$\dot{\alpha}_2 = -2i\omega \alpha_2$$

$$\dot{\alpha}_3 = -i\omega \alpha_1$$

(b) Differential equation for α_2: $\dot{\alpha}_2 = -2i\omega\alpha_2$

general solution: $\alpha_2 = B \exp\{-2i\omega t\}$

initial condition: $\alpha_2(t=0) = B = b$

$$\Rightarrow \quad \alpha_2 = b \exp\{-2i\omega t\}$$

(c) Define: $\alpha_+ = \alpha_1 + \alpha_3$

$$\frac{\partial \alpha_+}{\partial t} = \dot{\alpha}_1 + \dot{\alpha}_3 = -i\omega(\alpha_1 + \alpha_3) = -i\omega\alpha_+$$

Solution: $\alpha_+ = A\exp\{-i\omega t\} = \alpha_1 + \alpha_3 \quad \ldots \; (1)$

At $t=0$ $\qquad A = a + c$

Define : $\alpha_- = \alpha_1 - \alpha_3$

$$\frac{\partial \alpha_+}{\partial t} = \dot{\alpha}_1 - \dot{\alpha}_3 = i\omega(\alpha_1 - \alpha_3) = i\omega\alpha_-$$

Solution $\alpha_- = D\exp\{+i\omega t\} = \alpha_1 - \alpha_3 \quad \ldots\ldots (2)$

At $t=0$ $\qquad D = a - c$

Add (1) + (2) : $A\exp\{-i\omega t\} + D\exp(i\omega t\} = 2\alpha_1$

$$(a+c)\cos(\omega t) - i(a+c)\sin(\omega t)$$
$$+ (a-c)\cos(\omega t) + i(a-c)\sin(\omega t) = 2\alpha_1$$

$$a\cos(\omega t) - ic\sin(\omega t) = \alpha_1(t)$$

Subtract (1) − (2) : $A\exp\{-i\omega t\} - D\exp\{i\omega t\} = 2\alpha_3$

$$(a+c)\cos(\omega t) - i(a+c)\sin(\omega t)$$
$$-(a-c)\cos(\omega t) - (a-c)i\sin(\omega t) = 2\alpha_3$$

$$c\cos(\omega t) - ai\sin(\omega t) = \alpha_3(t)$$

$$\vec{\psi} = \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \alpha_3(t) \end{pmatrix} = \begin{pmatrix} a\cos(\omega t) - ic\sin(\omega t) \\ b\exp\{-2i\omega t\} \\ c\cos(\omega t) - ia\sin(\omega t) \end{pmatrix}$$

Which is same as for question 1.