PHYS 234: Quantum Physics 1 (Fall 2008)
Assignment 6 – Solutions

Issued: October 17, 2008 Due: 12.00pm, October 24, 2008

1. Given a mixture of atoms where 20% are in the state $\mid \downarrow_x \rangle$, 75% are in the state $\mid \uparrow_y \rangle$ and the rest are in the state $\mid \uparrow_z \rangle$, what is the statistical operator, ρ, for this mixture expressed as a matrix in the z-basis?

solution The generalised form for the statistical operator is given by the sum of products of relative weights and projector operators:

$$\begin{aligned}
\rho &= \sum P_i |i\rangle\langle i| \\
&= \frac{1}{5}\mid \downarrow_x \rangle\langle \downarrow_x \mid + \frac{3}{4}\mid \uparrow_y \rangle\langle \uparrow_y \mid + \frac{1}{20}\mid \uparrow_z \rangle\langle \uparrow_z \mid \\
&= \frac{1}{5}\frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}1 & -1\end{pmatrix} + \frac{3}{4}\frac{1}{2}\begin{pmatrix}1\\i\end{pmatrix}\begin{pmatrix}1 & -i\end{pmatrix} + \frac{1}{20}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} \\
&= \frac{1}{10}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix} + \frac{3}{8}\begin{pmatrix}1 & -i\\i & 1\end{pmatrix} + \frac{1}{20}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \\
&= \frac{1}{40}\begin{pmatrix}21 & -4-15i\\-4+15i & 19\end{pmatrix}
\end{aligned}$$

2. The spin state of an electron is represented (in the basis formed by the eigenvectors of σ_z) by the statistical operator

$$\begin{pmatrix}a & 0\\0 & b\end{pmatrix}$$

where a and b are real numbers with $a \geq 0,\ b \geq 0$ and $a + b = 1$

(a) If a x-Stern Gerlach measurement is made, what is the probability that the result obtained will be (a) deflected up, (b) deflected down?

solution For probability of deflected up:

$$\begin{aligned}
Prob(\mid \uparrow_x \rangle) &= Tr\{\rho \mid \uparrow_x \rangle\langle \uparrow_x \mid\} \\
&= \frac{1}{2}Tr\left\{\begin{pmatrix}a & 0\\b & 0\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\right\} \\
&= \frac{1}{2}Tr\left\{\begin{pmatrix}a & a\\b & b\end{pmatrix}\right\} = \frac{1}{2}(a+b) = \frac{1}{2}
\end{aligned}$$

For probability of deflected down:

$$\begin{aligned}
Prob(\mid \downarrow_x \rangle) &= Tr\{\rho \mid \downarrow_x \rangle\langle \downarrow_x \mid\} \\
&= \frac{1}{2}Tr\left\{\begin{pmatrix}a & 0\\b & 0\end{pmatrix}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}\right\} \\
&= \frac{1}{2}Tr\left\{\begin{pmatrix}a & -a\\-b & b\end{pmatrix}\right\} = \frac{1}{2}(a+b) = \frac{1}{2}
\end{aligned}$$

(b) Use these probabilities to compute the expectation value for this measurement.

solution

$$< \sigma_x >= (+1)Prob(\big| \uparrow_x \big\rangle) + (-1)Prob(\big| \downarrow_x \big\rangle) = \frac{1}{2} - \frac{1}{2} = 0$$

(c) Check that the result for (b) agrees with that calculated directly using the statistical operator formalism given by

$$< \sigma_x >= \text{Tr}\{\sigma_x \, \rho\}$$

solution

$$< \sigma_x >= \text{Tr}\{\sigma_x \, \rho\} = Tr\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right\} = Tr\left\{\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}\right\} = 0$$

3. Suppose a source emits silver atoms which are sent through three different types of Stern-Gerlach devices, each of them measuring with magnetic field gradients along the x, y and z axis. When the z or y measurements are performed, the probability that an atom is deflected up is $3/8$, while for the x-SG experiment the probability that an atom is deflected up is 3/4.

(a) What is the density operator (statistical operator) for this source?

solution In this case the information provided must be converted to a Bloch vector and used to construct the statistical operator:

$$\begin{aligned} < \sigma_x > &= (+1)Prob(\big| \uparrow_x \big\rangle) + (-1)Prob(\big| \downarrow_x \big\rangle) = Prob(\big| \uparrow_x \big\rangle) - (1 - Prob(\big| \uparrow_x \big\rangle)) \\ &= 2Prob(\big| \uparrow_x \big\rangle) - 1 = 2(3/4) - 1 = 1/2 \\ < \sigma_y > &= 2Prob(\big| \uparrow_y \big\rangle) - 1 = 2(3/8) - 1 = -1/4 \\ < \sigma_z > &= 2Prob(\big| \uparrow_z \big\rangle) - 1 = 2(3/8) - 1 = -1/4 \end{aligned}$$

Hence the Bloch vector is $\vec{s} = (< \sigma_x >, < \sigma_y >, < \sigma_z >)^T = (1/2, -1/4, -1/4)^T$, and the corresponding statistical operator is given by:

$$\rho = 1/2(\mathbb{1} + \vec{s} \cdot \vec{\sigma}) = \frac{1}{8}\begin{pmatrix} 3 & 2+i \\ 2-1 & 5 \end{pmatrix}$$

(b) Write down the measurement operator $\sigma_n = \vec{n}.\vec{\sigma}$ for a Stern Gerlach apparatus that measures along an axis rotated by +45° in the xy-plane.

solution The unit vector that defines the direction of measurement is

$$\vec{n} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Hence the measurement is described by the operator $\sigma_n = \vec{n}.\vec{\sigma}$, which is:

$$\sigma_n = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix}$$

(c) Compute the expectation value, $< \sigma_n >$.

solution The expectation value is given by:

$$< \sigma_n > = Tr\{\rho\sigma_n\} = Tr\left\{\frac{1}{8}\begin{pmatrix} 3 & 2+i \\ 2-1 & 5 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix}\right\}$$
$$= \frac{1}{8\sqrt{2}}Tr\left\{\begin{pmatrix} 1+3i & 3-3i \\ 5+5i & 1-3i \end{pmatrix}\right\} = \frac{1}{4\sqrt{2}}$$

Note that a very simple alternative result that we derived in class is that $< \sigma_n >= \vec{n} \cdot \vec{s}$

(d) What is the probability that atoms are deflected up in this experiment?

solution

$$< \sigma_n >= (+1)Prob(| \uparrow_n \rangle) + (-1)Prob(| \downarrow_n \rangle) = 2Prob(| \uparrow_n \rangle) - 1 = \frac{1}{4\sqrt{2}}$$

Hence

$$Prob(| \uparrow_n \rangle = \frac{1}{2}\left(\frac{1}{4\sqrt{2}} + 1\right)$$