PHYS 234: Quantum Physics 1 (Fall 2008)
Assignment 5

Issued: October 10, 2008 Due: 12.00pm, October 17, 2008

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1. Consider source atoms for a Stern Gerlach measurement which are in the state described by

$$| \rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1+i \\ 2 \end{pmatrix}$$

(a) Calculate the respective probabilities for the atoms to be deflected up and down in a z-SG experiment.

solution For this state, the coefficients of $| \uparrow_z \rangle$ and $| \downarrow_z \rangle$ are easy to pick off:

$$\alpha = \frac{1+i}{\sqrt{6}} \Rightarrow Prob(| \uparrow_z \rangle) = |\alpha|^2 = 1/3$$

$$\beta = \frac{2}{\sqrt{6}} \Rightarrow Prob(| \downarrow_z \rangle) = |\alpha|^2 = 2/3$$

(b) Calculate the respective probabilities for the atoms to be deflected up and down in an x-SG experiment.

solution Decompose into $| \uparrow_x \rangle$ and $| \downarrow_x \rangle$

$$| \rangle = | \uparrow_x \rangle \gamma + | \downarrow_x \rangle \delta$$

The coefficients and probabilities are then:

$$\gamma = \langle \uparrow_x | \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \frac{1}{\sqrt{6}} \begin{pmatrix} 1+i \\ 2 \end{pmatrix} \Longrightarrow Prob(| \uparrow_x \rangle) = |\gamma|^2 = 5/6$$

$$\delta = \langle \downarrow_x | \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \end{pmatrix} \frac{1}{\sqrt{6}} \begin{pmatrix} 1+i \\ 2 \end{pmatrix} \Longrightarrow Prob(| \downarrow_x \rangle) = |\delta|^2 = 1/6$$

(c) Using the operator σ_z or otherwise, calculate the expectation value for the z-SG experiment.

solution

$$< \sigma_z > = \langle | \sigma_z | \rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1+i & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1+i \\ 2 \end{pmatrix} = -1/3$$

(d) Using the operator σ_x or otherwise, calculate the expectation value for the x-SG experiment.

solution

$$< \sigma_x > = \langle | \sigma_x | \rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1+i & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1+i \\ 2 \end{pmatrix} = 2/3$$

2. An electron is in the spin state

$$|\;\rangle = A\begin{pmatrix}3i\\4\end{pmatrix}$$

(a) Determine the normalisation constant A.

solution For the state vector to be normalised the norm must be equal to 1

$$\|\;|\;\rangle\;\| = \sqrt{\langle|\rangle} = |A|^2(9+16) = 25|A|^2 = 1 \Rightarrow A = 1/5$$

(b) Find the expectation values for σ_x, σ_y,σ_z.

solution

$$<\sigma_x> = \langle\;|\sigma_x|\;\rangle = \frac{1}{25}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}-3i\\4\end{pmatrix} = \frac{1}{25}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}4\\3i\end{pmatrix} = \frac{1}{25}(-12i+12i) = 0$$

$$<\sigma_y> = \langle\;|\sigma_y|\;\rangle = \frac{1}{25}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\begin{pmatrix}-3i\\4\end{pmatrix} = \frac{1}{25}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}-4i\\-3\end{pmatrix} = \frac{1}{25}(-12-12) = \frac{-24}{25}$$

$$<\sigma_z> = \langle\;|\sigma_z|\;\rangle = \frac{1}{25}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}-3i\\4\end{pmatrix} = \frac{1}{25}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}3i\\-4\end{pmatrix} = \frac{1}{25}(9-16) = \frac{-7}{25}$$

(c) Find the "uncertainties" $\Delta\sigma_x$, $\Delta\sigma_y$, $\Delta\sigma_z$, where $\Delta X^2 =< X^2 > - < X >^2$

solution The operators $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbb{1}$, hence the expectation values are

$$<\sigma_x^2> = \langle\;|\sigma_x^2|\;\rangle = \langle\;|\mathbb{1}|\;\rangle = \langle|\rangle = 1 =<\sigma_z^2>=<\sigma_y^2>$$

The uncertainty, $\Delta\sigma_x$, is therefore given by

$$\Delta\sigma_x = \sqrt{<\sigma_x^2> - <\sigma_x>^2} = \sqrt{1-0} = 1$$

Similarly

$$\Delta\sigma_y = \sqrt{1-\left(\frac{-24}{25}\right)^2} = \frac{7}{25}$$

and

$$\Delta\sigma_z = \sqrt{1-\left(\frac{-7}{25}\right)^2} = \frac{24}{25}$$

3. Express the operator $A = |\uparrow_x\rangle\langle\uparrow_z| + |\downarrow_x\rangle\langle\downarrow_z|$ as a linear function of $\vec{\sigma}$. What is A^2 ?

solution There is a brute-force way to approach this question where one substitutes the numeric vectors (in the z-basis), calculates the resulting matrix and then rewrites this in terms of Pauli operators. However, the solution can be found by keeping the problem in bra-ket notation.

$$\begin{aligned}|\uparrow_x\rangle\langle\uparrow_z| + |\downarrow_x\rangle\langle\downarrow_z| &= \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle + |\downarrow_z\rangle\right)\langle\uparrow_z| + \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle - |\downarrow_z\rangle\right)\langle\downarrow_z| \\ &= \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle\langle\downarrow_z| + |\downarrow_z\rangle\langle\uparrow_z| + |\uparrow_z\rangle\langle\uparrow_z| - |\downarrow_z\rangle\langle\downarrow_z|\right) \\ &= \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z)\end{aligned}$$

For the expression of σ_x in terms of $|\uparrow_z\rangle$'s see courseware notes equation (2.9.6) on page 41. For A^2 we get

$$\left(\frac{1}{\sqrt{2}}(\sigma_x+\sigma_z)\right)^2 = \frac{1}{2}(\sigma_x^2+\sigma_x\sigma_z+\sigma_z\sigma_x+\sigma_z^2) = \frac{1}{2}(1+0+1) = 1$$

The important point here is that the order of the operators is maintained and the anticyclic properties of Paulis is used.

4. Suppose you have a state

$$|\Psi\rangle = \sqrt{\frac{3}{5}}|\uparrow_z\rangle + \sqrt{\frac{2}{5}}|\downarrow_z\rangle$$

What is expectation value for a Stern Gerlach experiment with the field gradient in the x-direction ?

solution Again there is more than one approach here, and the brute-force method is certainly one option. The simplest, I think is to decompose each of the terms into "$\pm x$ atoms".

$$\begin{aligned}|\Psi\rangle &= \sqrt{\frac{3}{5}}|\uparrow_z\rangle + \sqrt{\frac{2}{5}}|\downarrow_z\rangle \\ &= \sqrt{\frac{3}{5}}\frac{1}{\sqrt{2}}\left(|\uparrow_x\rangle + |\downarrow_x\rangle\right) + \sqrt{\frac{2}{5}}\frac{1}{\sqrt{2}}\left(|\uparrow_x\rangle - |\downarrow_x\rangle\right) \\ &= \frac{\sqrt{3}+\sqrt{2}}{\sqrt{10}}|\uparrow_x\rangle + \frac{\sqrt{3}-\sqrt{2}}{\sqrt{10}}|\downarrow_x\rangle\end{aligned}$$

Hence the respective probabilities for up and down outcomes are:

$$Prob(|\uparrow_x\rangle = \left(\frac{\sqrt{3}+\sqrt{2}}{\sqrt{10}}\right)^2, \text{ and } Prob(|\downarrow_x\rangle = \left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{10}}\right)^2$$

The expectation value is then the sum of the products of eigenvalues and probabilities:

$$<\sigma_x>= \left(\frac{\sqrt{3}+\sqrt{2}}{\sqrt{10}}\right)^2 - \left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{10}}\right)^2 = \frac{2}{5}\sqrt{6}$$

Alternatively you can compute either $\langle\Psi|\sigma_x|\Psi\rangle$ or $\mathrm{Tr}\{\sigma_x|\Psi\rangle\langle\Psi|\}$.