PHYS 234: Quantum Physics 1 (Fall 2008)
Assignment 4 – SOLUTIONS

Issued: October 3, 2008 Due: 12.00pm, October 10, 2008

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1. Show that one gets a 50%:50% probability for deflection "up" and "down" in a y-Stern Gerlach measurement when atoms that are preselected as "+x atoms" or "-x atoms" are used as a source.

solution Decompose "+x atoms" and "-x atoms" into "$\pm y$ atoms"
For "+x atoms"

$$\begin{aligned}
|\uparrow_x\rangle &= |\uparrow_y\rangle\langle\uparrow_y|\uparrow_x\rangle + |\downarrow_y\rangle\langle\downarrow_y|\uparrow_x\rangle \\
&= |\uparrow_y\rangle\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} + |\downarrow_y\rangle\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \\
&= |\uparrow_y\rangle\frac{1}{2}(1-i) + |\downarrow_y\rangle\frac{1}{2}(1+i)
\end{aligned}$$

For "-x atoms"

$$\begin{aligned}
|\downarrow_x\rangle &= |\uparrow_y\rangle\langle\uparrow_y|\downarrow_x\rangle + |\downarrow_y\rangle\langle\downarrow_y|\downarrow_x\rangle \\
&= |\uparrow_y\rangle\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} + |\downarrow_y\rangle\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \\
&= |\uparrow_y\rangle\frac{1}{2}(1+i) + |\downarrow_y\rangle\frac{1}{2}(1-i)
\end{aligned}$$

The respective probabilities are given by the squared values of the state coefficients, which in both cases show 50%:50% probabilities:

$$\left|\frac{1}{2}(1+i)\right|^2 = \left|\frac{1}{2}(1-i)\right|^2 = \frac{1}{2}$$

2. Compare the following situations.

   (1) A beam of atoms, half of which are preselected as "+z atoms", the other half as "-z atoms", is sent through a Stern Gerlach apparatus that

   (i) sorts in the z direction

   (ii) sorts in the x direction

   (2) A beam of atoms, all of which are preselected as "+x atoms", is sent through Stern Gerlach apparatus that

   (i) sorts in the z direction

   (ii) sorts in the x direction

   Do the measurement results from (1) and (2) enable you to tell situations (1) and (2) apart. That is, compare the combined results of the measurements (1)(i) and (1)(ii) with those of (2)(i) and (2)(ii) and see if they are different.

solution We can only distinguish the beams in (1) and (2) on the basis of the measurements made in (i) and (ii) respectively:
Beam (1) "$\pm z$ atoms"

(i) In z direction:

$$\begin{aligned} Prob(\big|\uparrow_z\big\rangle) &= Prob(\text{choose}\big|\uparrow_z\big\rangle)\times Prob(\text{result}\big|\uparrow_z\big\rangle)+Prob(\text{choose}\big|\downarrow_z\big\rangle)\times Prob(\text{result}\big|\uparrow_z\big\rangle) \\ &= \frac{1}{2}\times 1+\frac{1}{2}\times 0=\frac{1}{2} \end{aligned}$$

$$\begin{aligned} Prob(\big|\downarrow_z\big\rangle) &= Prob(\text{choose}\big|\uparrow_z\big\rangle)\times Prob(\text{result}\big|\downarrow_z\big\rangle)+Prob(\text{choose}\big|\downarrow_z\big\rangle)\times Prob(\text{result}\big|\downarrow_z\big\rangle) \\ &= \frac{1}{2}\times 0+\frac{1}{2}\times 1=\frac{1}{2} \end{aligned}$$

(ii) In x direction:

$$\begin{aligned} Prob(\big|\uparrow_x\big\rangle) &= Prob(\text{choose}\big|\uparrow_z\big\rangle)\times Prob(\text{result}\big|\uparrow_x\big\rangle)+Prob(\text{choose}\big|\downarrow_z\big\rangle)\times Prob(\text{result}\big|\uparrow_x\big\rangle) \\ &= \frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}=\frac{1}{2} \end{aligned}$$

$$\begin{aligned} Prob(\big|\downarrow_x\big\rangle) &= Prob(\text{choose}\big|\uparrow_z\big\rangle)\times Prob(\text{result}\big|\downarrow_x\big\rangle)+Prob(\text{choose}\big|\downarrow_z\big\rangle)\times Prob(\text{result}\big|\downarrow_x\big\rangle) \\ &= \frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}=\frac{1}{2} \end{aligned}$$

Beam (2) "$+x$ atoms"

(i) In z direction:

$$Prob(\big|\uparrow_z\big\rangle)=\frac{1}{2}$$

$$Prob(\big|\downarrow_z\big\rangle)=\frac{1}{2}$$

(ii) In x direction:

$$Prob(\big|\uparrow_x\big\rangle)=1$$

$$Prob(\big|\downarrow_x\big\rangle)=0$$

So we see that the results from (i) z Stern Gerlach are the same ($\frac{1}{2},\frac{1}{2}$), while the results from (ii) x Stern Gerlach are different: Beam (1) gives ($\frac{1}{2},\frac{1}{2}$), while Beam (2) gives (1,0). Ultimately, we can distinguish between the two situations.

3. A source produces atoms which using the z-basis are described as $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$ which are moving along the y-axis. A magnetic field is applied to the atoms in the $+y$-direction, which would rotate the (classical) dipole moment by an angle $\phi=\pi/2$. The rotation matrix is given by

$$R_y(\phi)=\begin{pmatrix}\cos(\phi/2) & \sin(\phi/2)\\ -\sin(\phi/2) & \cos(\phi/2)\end{pmatrix}$$

(a) Calculate the mathematical description of the state after the rotation.

solution

$$R_y(\pi/2)\big|\uparrow_y\big\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1+i\\ -1+i\end{pmatrix}$$

(b) If the rotated atoms are now sent into a Stern Gerlach apparatus directed along the z-direction, calculate the probabilities that the atoms are deflected up and down with respect to the z-axis.

solution Decompose into "$\pm z$ atoms"

$$\big| = \big\rangle \big|\uparrow_z\big\rangle\delta_z + \big|\downarrow_z\big\rangle\gamma_z$$

with

$$\delta_z = \big\langle\uparrow_z\big|\big\rangle = \begin{pmatrix}1 & 0\end{pmatrix}\frac{1}{2}\begin{pmatrix}1+i\\ -1+i\end{pmatrix} = \frac{1}{2}(1+i)$$

$$\gamma_z = \big\langle\downarrow_z\big|\big\rangle = \begin{pmatrix}0 & 1\end{pmatrix}\frac{1}{2}\begin{pmatrix}1+i\\ -1+i\end{pmatrix} = \frac{1}{2}(-1+i)$$

Hence:
Prob($\big|\uparrow_z\big\rangle$) $= |\delta_z|^2 = \frac{1}{4}(1+i)(1-i) = \frac{1}{2}$
Prob($\big|\downarrow_z\big\rangle$) $= |\gamma_z|^2 = \frac{1}{4}(-1+i)(-1-i) = \frac{1}{2}$

(c) If the rotated atoms are sent into a y-Stern Gerlach apparatus, calculate the probabilities for the two possible outcomes in this case.

solution Decompose into "$\pm y$ atoms"

$$\big| = \big\rangle \big|\uparrow_y\big\rangle\delta_y + \big|\downarrow_y\big\rangle\gamma_y$$

with

$$\delta_y = \big\langle\uparrow_y\big|\big\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}\frac{1}{2}\begin{pmatrix}1+i\\ -1+i\end{pmatrix} = \frac{1}{2\sqrt{2}}(1+i+i+1) = \frac{1}{\sqrt{2}}(1+i)$$

$$\gamma_y = \big\langle\downarrow_y\big|\big\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\end{pmatrix}\frac{1}{2}\begin{pmatrix}1+i\\ -1+i\end{pmatrix} = \frac{1}{2\sqrt{2}}(1+i-i-1) = 0$$

Hence:
Prob($\big|\uparrow_y\big\rangle$) $= |\delta_y|^2 = \frac{1}{2}(1+i)(1-i) = 1$
Prob($\big|\downarrow_y\big\rangle$) $= |\gamma_y|^2 = 0$

(d) Interpret the results of your analysis for these two Stern Gerlach experiments.

solution Because the original atom is $\big|\uparrow_y\big\rangle$, when it is rotated around the y-axis, it does not change (although mathematically it does look different). Hence, when put into a z-SG half the atoms are deflected up and half down, while in a y-SG all the atoms are deflected up.

4. (Problem A.25) Let

$$\mathbf{T} = \begin{pmatrix}1 & 1-i\\ 1+i & 0\end{pmatrix}$$

(a) Verify that $\mathbf{T}$ is Hermitian.

solution

$$T^\dagger = \tilde{T}^* = \begin{pmatrix} 1 & 1-i \\ 1+i & 0 \end{pmatrix}$$

(b) Find its eigenvalues (note that they are real).

solution

$$\begin{vmatrix} (1-\lambda) & (1-i) \\ (1+i) & (0-\lambda) \end{vmatrix} = -(1-\lambda)\lambda - 1 - 1 = 0$$

$$\lambda^2 - \lambda - 2 = 0 \qquad \Rightarrow \qquad \lambda_1 = 2; \ \ \lambda_2 = -1$$

(c) Find and normalise the eigenvectors and verify that they are orthogonal.

solution

$$\begin{pmatrix} 1 & 1-i \\ 1+i & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 2 \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \ \Rightarrow \ \alpha + (1-i)\beta = 2\alpha \ \Rightarrow \ \alpha = (1-i)\beta$$

choose $\beta = 1, \alpha = (1-i)$ and normalise, hence

$$|\,\lambda_1\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1-i \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1-i \\ 1+i & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = - \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \ \Rightarrow \ \alpha + (1-i)\beta = -\alpha \ \Rightarrow \ \alpha = -\frac{1}{2}(1-i)\beta$$

choose $\beta = 2, \alpha = (i-1)$ and normalise, hence

$$|\,\lambda_2\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} i-1 \\ 2 \end{pmatrix}$$

(d) Construct the unitary diagonalising matrix $\mathbf{S}$, and check explicitly that it diagonalises $\mathbf{T}$.

solution Form the matrix $\mathbf{S}^{-1}$ from using the eigenvectors as columns:

$$\mathbf{S}^{-1} = \frac{1}{\sqrt{3}} \begin{pmatrix} (1-i) & (i-1)/\sqrt{2} \\ 1 & \sqrt{2} \end{pmatrix}$$

Since $\mathbf{S}^{-1}$ is unitary, then the inverse is the adjoint matrix

$$\mathbf{S} = (\mathbf{S}^{-1})^\dagger = \frac{1}{\sqrt{3}} \begin{pmatrix} (1+i) & 1 \\ (-i-1)/\sqrt{2} & \sqrt{2} \end{pmatrix}$$

Now check that these matrices diagonalise $\mathbf{T}$

$$\begin{aligned} \mathbf{STS}^{-1} &= \frac{1}{3} \begin{pmatrix} (1+i) & 1 \\ (-i-1)/\sqrt{2} & \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 1-i \\ 1+i & 0 \end{pmatrix} \begin{pmatrix} (1-i) & (i-1)/\sqrt{2} \\ 1 & \sqrt{2} \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} (1+i) & 1 \\ (-i-1)/\sqrt{2} & \sqrt{2} \end{pmatrix} \begin{pmatrix} 2(1-i) & (1-i)/\sqrt{2} \\ 2 & -\sqrt{2} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 6 & 0 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

(e) Check that det($\mathbf{T}$) and Tr($\mathbf{T}$) are the same for $\mathbf{T}$ as they are for its diagonalised form.

solution

$$\text{Tr}(\mathbf{T}) = 1 + 0 = 1 \ \text{ and } \ \text{Tr}(\mathbf{STS}^{-1}) = 2 - 1 = 1$$

$$\det(\mathbf{T}) = 0 - (1+i)(1-i) = -2 \ \text{ and } \ \det(\mathbf{STS}^{-1}) = 2 \times (-1) = -2$$