PHYS 234: Quantum Physics 1 (Fall 2008)
Assignment 3 – Solutions

Issued: September 26, 2008 Due: 12.00pm, October 3, 2008

Pauli Matrices (See sections 2.9-2.10 of Section 2 of the UW-Courseware)
Pauli operators in coordinate representation in the basis $|\uparrow_z\rangle, |\downarrow_z\rangle$:

$$\sigma_x \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \sigma_y \rightarrow \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \sigma_z \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

1. The Pauli matrices along with the (2x2) Identity matrix form the basic building blocks for any (2x2) matrix. This can be demonstrated in the following way using an extension of vector notation.

   We define a vector of (complex) scalars, $\vec{a} = (a_x, a_y, a_z)$ and a vector of Pauli matrices, $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. These can be combined in a dot product, $\vec{a}.\vec{\sigma} = a_x\sigma_x + a_y\sigma_y + a_z\sigma_z$.

   Using the (complex) scalar a_0, any (2x2) matrix, $\hat{T}$, may be expressed:

$$\hat{T} = a_0 \mathbb{1} + \vec{a}.\vec{\sigma}$$

   (a) Use this result to find expressions for a_0 and the components of $\vec{a}$ in terms of the matrix elements, T_{ij}, of $\hat{T}$.

   [*Hint*: Write $\hat{T}$ in matrix form and equate it, element by element, to the expanded form of the RH-side of the equation above to give four coupled equations. Rearrange these to find a_0, a_x, a_y, a_z in terms of T_{ij}.]

Solution

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = a_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + a_y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + a_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hence

$$T_{11} = a_0 + a_z, \qquad T_{21} = a_x + ia_y$$

$$T_{22} = a_0 - a_z, \qquad T_{12} = a_x - ia_y$$

which can be solved to give

$$a_0 = 1/2(T_{11} + T_{22}), \qquad a_x = 1/2(T_{21} + T_{12})$$

$$a_z = 1/2(T_{11} - T_{22}), \qquad a_y = -i/2(T_{21} - T_{12})$$

   (b) Use these equations to find a_0, a_x, a_y, a_z for the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

solution

$$a_0 = 1/2(0+0) = 0, \qquad a_x = 1/2(0+1) = 1/2$$

$$a_z = 1/2(0+0) = 0, \qquad a_y = -i/2(0-1) = i/2$$

Now check explicitly that this is equivalent to

$$a_0 \mathbb{1} + a_x\sigma_x + a_y\sigma_y + a_z\sigma_z$$

solution

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{i}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

2. Show explicitly that $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\mathbb{1}$ where $i, j = x, y, z$. In other words, show that for $i = j$, $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbb{1}$, while for $i \neq j$, show $\{\sigma_i, \sigma_j\} = 0$, where curly brackets indicate the anticommutator: $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$. (You don't need to evaluate all permutations but a sufficient number to convince yourself that this statement is true)

solution In each case for $i = j$

$$\sigma_x^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_y^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

while for $i \neq j$, for example

$$\sigma_x\sigma_z + \sigma_z\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 0$$

What you see here is first that $(\sigma_x\sigma_z) = -(\sigma_z\sigma_x)$, and if you look closely that $(\sigma_x\sigma_z) = -i\sigma_y$ and therefore that $(\sigma_z\sigma_x) = i\sigma_y$

3. Using the fundamental algebraic properties in eq (2.9.11), calculate the following expressing each as $a_0 \mathbb{1} + \vec{a}.\vec{\sigma}$

solution

$$\left[\frac{1}{2}(\mathbb{1} + \sigma_x)\right]^2 = \frac{1}{2}(\mathbb{1} + \sigma_x)$$

$$\left[\frac{1}{2}(\mathbb{1} - \sigma_x)\right]^2 = \frac{1}{2}(\mathbb{1} - \sigma_x)$$

$$\frac{1}{2}(\mathbb{1} + \sigma_x)\frac{1}{2}(\mathbb{1} - \sigma_x) = \frac{1}{2}\mathbb{1}$$

$$(i\sigma_z)\sigma_x(-i\sigma_z) = -\sigma_x$$

$$(i\sigma_z)\sigma_y(-i\sigma_z) = -\sigma_y$$

$$\sqrt{\frac{1}{2}}(\mathbb{1} + i\sigma_z)\sigma_x\sqrt{\frac{1}{2}}(\mathbb{1} - i\sigma_z) = -\sigma_y$$

$$\sqrt{\frac{1}{2}}(\mathbb{1} + i\sigma_z)\sigma_y\sqrt{\frac{1}{2}}(\mathbb{1} - i\sigma_z) = \sigma_x$$