# PHYS 234: Quantum Physics 1 (Fall 2008)
# Assignment 10 – Solutions

Issued: November 21, 2008 Due: 12.00pm, November 28, 2008

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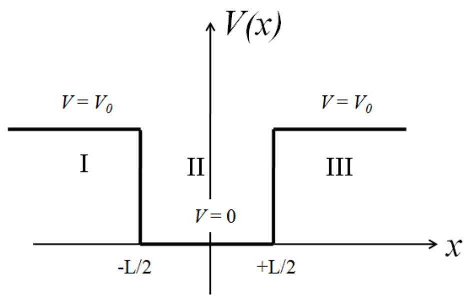
**Finite Square Potential Well**

Consider the potential function shown in the figure below. This is a square well with the potential inside the well equal to zero and the potential outside the well equal to a constant and finite value, $V_0$.




1. **Bound States,** $E < V_0$

   (a) **Outside the well (Regions I and III)**

      i. Write down the Time Independent Schroedinger Equation for the regions outside the well (same for both regions)

**solution**

$$\text{TISE} \quad \frac{d^2\psi(x)}{dx^2} = k_I^2\psi(x) \quad \text{where} \quad k_I = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

Note how the TISE has been manipulated to keep $k_I$ real and hence $k_I^2$ positive

      ii. Write down the general solutions for regions I and III.

**solution**

$$\text{Region I:} \quad \psi_I(x) = A\exp\{-k_I x\} + B\exp\{+k_I x\}$$

$$\text{Region III:} \quad \psi_{III}(x) = F\exp\{-k_I x\} + G\exp\{+k_I x\}$$

      iii. For the equation in each region use arguments related to ensuring the wavefunction is normalisable to eliminate terms in the general solution.

**solution** For $\psi_I(x)$, which is defined only for $x < -L/2$, the term $A\exp\{-k_I x\}$ will diverge as $x$ goes to $-\infty$. Hence $A = 0$
For $\psi_{III}(x)$, which is defined only for $x > +L/2$, the term $G\exp\{+k_I x\}$ will diverge as $x$ goes to $+\infty$. Hence $G = 0$
Consequently, the solutions become:

$$\psi_I(x) = B\exp\{+k_I x\} \qquad x < -L/2$$

$$\psi_{III}(x) = F\exp\{-k_I x\} \qquad x > +L/2$$

iv. Use symmetry arguments to divide your solutions on the basis of symmetry and consequently to further reduce the total number of constants across both solutions to one.

**solution** Because the potential is symmetric about its centre line, the solutions will be either even or odd functions:

$$\text{EVEN:} \quad \psi(x) = \psi(-x) \quad \Rightarrow \quad B = F$$

$$\text{ODD:} \quad \psi(-x) = -\psi(x) \quad \Rightarrow \quad B = -F$$

Hence solutions outside well for $(E < V_0)$ are:

EVEN: $\psi_I(x) = B\exp\{+k_I x\}$ $\quad x < -L/2$
$\psi_{III}(x) = B\exp\{-k_{III} x\}$ $\quad x > +L/2$

ODD: $\psi_I(x) = B\exp\{+k_I x\}$ $\quad x < -L/2$
$\psi_{III}(x) = -B\exp\{-k_{III} x\}$ $\quad x > +L/2$

(b) **Inside the well (Region II)**

i. Write down the Time Independent Schroedinger Equation for this region of the $x$-axis

**solution**

$$\text{TISE} \quad \frac{d^2\psi(x)}{dx^2} = -k_{II}^2\psi(x) \quad \text{where} \quad k_{II} = \frac{\sqrt{2m(E)}}{\hbar}$$

Note how the TISE has been (perhaps less obviously) manipulated to keep $k_I$ real and hence $k_I^2$ positive.

ii. Use symmetry arguments to show that the solutions should be alternately sine and cosine functions and establish the solution is each case.

**solution**

$$\text{General solution:} \quad \psi_{II}(x) = C\sin(k_{II}x) + D\cos(k_{II}x) \qquad -L/2 < x < +L/2$$

As before, the symmetry of the potential means that the eigenfunctions will be alternately even and odd functions. We can use this concept to simplify the general solutions into two groups:

$$\text{EVEN:} \quad \psi_{II}(x) = D\cos(k_{II}x) \quad \Rightarrow \quad C = 0$$

$$\text{ODD:} \quad \psi_{II}(x) = C\sin(k_{II}x) \quad \Rightarrow \quad D = 0$$

iii. Which type of function will be associated with the lowest energy and why?

**solution** The solution with the lowest energy will have zero nodes. The only function that will satisfy this condition is the cosine function (Even function).

Your combined solutions for regions I, II and III should now resemble those in the table on page 156 of the text book.

| Even Solutions | Odd Solutions | Region |
|---|---|---|
| $\psi_I = Ae^{\alpha x}$ | $\psi_I = Ce^{\alpha x}$ | I: $x < -L/2$ |
| $\psi_{II} = B\cos kx$ | $\psi_{II} = B\sin kx$ | II: $-L/2 < x < +L/2$ |
| $\psi_{III} = Ae^{-\alpha x}$ | $\psi_{III} = -Ce^{-\alpha x}$ | III: $x > +L/2$ |

(c) Select the even set of functions. Apply the boundary conditions at $x = +L/2$ and derive a formula for the allowed energies of the bound state even eigenfunctions of this potential well.

**solution** Boundary conditions at $x = +L/2$

$$\psi(+L/2) \quad \Rightarrow \quad B\exp\{-k_I L/2\} = D\cos(k_{II}L/2)$$

$$\left.\frac{d\psi(x)}{dx}\right|_{+L/2} \quad \Rightarrow \quad -k_I B\exp\{-k_I L/2\} = -k_{II} D\sin(k_{II}L/2)$$

Dividing these equations:

$$k_I = k_{II}\tan(k_{II}L/2)$$

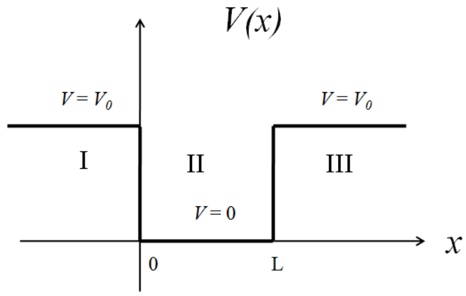
Substitute for $k_I$ and $k_{II}$

$$\sqrt{2m(V_0 - E)} = \sqrt{2mE}\tan\left(\frac{L\sqrt{2mE}}{2\hbar}\right)$$

This transcendental equation can be solved graphically or numerically to find the eigenenergies for a given potential well height $V_0$.

2. **Scattering States,** $E > V_0$

For the analysis of the scattering states the potential well is shifted on the $x$-axis to lie between $x = 0$ and $x = L$. Now consider a particle incident from $x = -\infty$ in the direction of increasing $x$.




(a) Write down the Time Independent Schroedinger Equation and its general solution in each region of the $x$-axis.

**solution** The TISE has the same form in each region:

$$\frac{d^2\psi(x)}{dx^2} = -k^2\psi(x) \quad \text{where} \quad k = \frac{\sqrt{2m(E - V(x))}}{\hbar}$$

The general solutions in each region are therefore:

$$\begin{aligned}
\text{region I:} \quad & \psi_I(x) = A_0 \exp\{ik_I x\} + A \exp\{-ik_I x\} \quad \text{where} \quad k_I = \frac{\sqrt{2m(E - V_0)}}{\hbar} \\
\text{region II:} \quad & \psi_{II}(x) = B \exp\{ik_{II} x\} + C \exp\{-ik_{II} x\} \quad \text{where} \quad k_{II} = \frac{\sqrt{2mE}}{\hbar} \\
\text{region III:} \quad & \psi_{III}(x) = D \exp\{ik_I x\} + F \exp\{-ik_I x\} \quad \text{where} \quad k_I = \frac{\sqrt{2m(E - V_0)}}{\hbar}
\end{aligned}$$

(b) If possible, interpret each term in these general solutions in a physical sense. Eliminate those terms that cannot be justified.

**solution** All solutions are plane waves that are interpreted as follows:

region I: $A_0$ is incident wave, $A$ is the wave reflected at $x = 0$
region II: $B$ is wave transmitted beyond $x = 0$, $C$ is the wave reflected at $x = L$
region III: $D$ is wave transmitted beyond $x = L$

However, there is nothing in the description of the potential beyond $x = L$ to reflect a wave back, so the coefficient $F = 0$

(c) Apply the appropriate boundary conditions at $x = 0$ and $x = L$ to match the solutions from individual regions. This will result in four equations containing five unknown constants.

**solution** Applying the conditions for continuity of $\psi(x)$ and $d\psi/dx$ at $x = 0$ and $x = L$ results in the following four equations:

$$A_0 + A = B + C \tag{1}$$
$$ik_I A_0 - ik_I A = ik_{II} B - ik_{II} C \tag{2}$$
$$B \exp\{ik_{II} L\} + C \exp\{-ik_{IIL}\} = D \exp\{ik_I L\} \tag{3}$$
$$ik_{II} B \exp\{ik_{II} L\} - ik_{II} C \exp\{-ik_{IIL}\} = ik_I D \exp\{ik_I L\} \tag{4}$$

(d) Analyse this set of equations to the point where the you can write the transmitted amplitude in terms of the incident amplitude and hence compute the Transmission coefficient for traversing the potential well.

**solution** For transmission through the entire well we require $D/A_0$
Use (1) and (2) to eliminate $A$:

$$2k_I A_0 = (k_{II} + k_I)B - (k_{II} - k_I)C \tag{5}$$

Use (3) and (4) to express $B$ and $C$ in terms of $D$:

$$B = \frac{k_{II} + k_I}{2k_{II}} D \exp\{ik_I L\} \exp\{-ik_{II} L\} \tag{6}$$

$$C = \frac{k_{II} - k_I}{2k_{II}} D \exp\{ik_I L\} \exp\{ik_{II} L\} \tag{7}$$

Substitute (6) and (7) into (5)

$$4k_I k_{II} A_0 = \left[(k_{II} + k_I)^2 \exp\{-ik_{II} L\} - (k_{II} - k_I)^2 \exp\{ik_{II} L\}\right] D \exp\{ik_I L\}$$

Since incident and transmitted eigenfunctions are plane waves, and the velocity on either side of the well is the same:

$$\begin{aligned} T &= \frac{|D|^2}{|A_0|^2} \\ &= \frac{4k_I^2 k_{II}^2}{4k_I^2 k_{II}^2 + (k_I^2 - k_{II}^2)^2 \sin^2(k_{II} L)} \end{aligned}$$