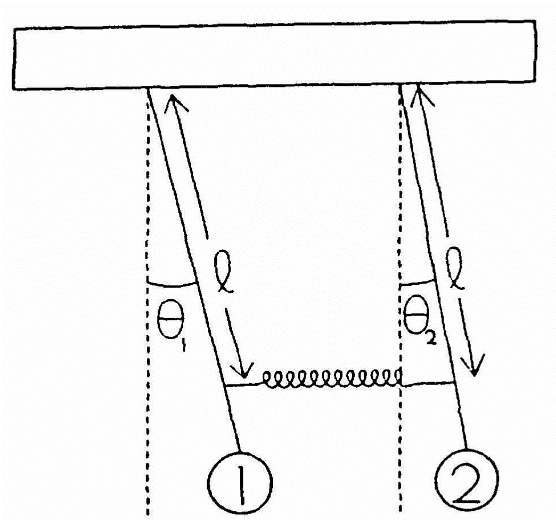
THE UNIVERSITY OF WATERLOO

Physics 360 – Experiment 5
COUPLED OSCILLATORS

References: Mechanics, A. Sommerfeld
Theoretical Mechanics, R.A. Becker

The theory of coupled oscillators is very basic to the radiation and absorption of energy. Molecules, atoms and nuclei are oscillating systems, and are coupled to oscillating electro magnetic fields through their electric or magnetic dipole or other moments. In solids, the coupling and masses of adjacent atoms determines the frequency spectrum of the sound waves which can propagate through them.



In this experiment the oscillators are pendulums and the effects of coupling two pendulums are studied. We begin by writing the general equation of motion for a single simple pendulum,

$$I\ddot{\theta} = -\Gamma \tag{1}$$

where I is the moment of inertia of the pendulum and Γ is the restoring torque. If the string has no mass and the masses are considered point charges we can write

$$mL^2\ddot{\theta} = -(mg\sin\theta)L \tag{2}$$

where L is the total pendulum length. This equation is difficult to solve because the angle θ appears on the right hand side as a trigonometric argument. However, if we keep the angles small we can write to a good approximation $\sin\theta \cong \theta$ and Eq. 2 becomes

$$\ddot{\theta} = -\frac{g}{L}\theta \tag{3}$$

with solutions $\theta = \theta_0 \cos(\omega_0 t)$ (4)

where, after direct substitution, we find that the angular frequency $\omega_0 = \sqrt{g/L}$.

If we now consider two coupled pendulums, it is apparent that for $\theta_2 > \theta_1$ the coupling torque acting on pendulum 1 by 2 is given by,

$$\Gamma_{21} = (k\ell\sin\theta_2)\ell - (k\ell\sin\theta_1)\ell = k\ell^2(\sin\theta_2 - \sin\theta_1) \tag{5}$$

where *l* indicates where the spring is attached. Assuming the spring has no mass we can then write for pendulum 1

$$mL^2\ddot{\theta}_1 = -mgL\sin\theta_1 + k\ell^2(\sin\theta_2 - \sin\theta_1) \tag{6}$$

which for small displacements becomes

$$\ddot{\theta}_1 = -\frac{g}{L}\theta_1 + \frac{k\ell^2}{mL^2}(\theta_2 - \theta_1) \tag{7a}$$

and similarly for pendulum 2

$$\ddot{\theta}_2 = -\frac{g}{L}\theta_2 + \frac{k\ell^2}{mL^2}(\theta_1 - \theta_2) \tag{7b}$$

Two important special cases, the "normal" modes, can be studied. The even mode is defined by $\theta_1 = \theta_2$ and $\dot{\theta}_1 = \dot{\theta}_2$ at t = 0. The pendulums will in effect not be coupled, since the spring extension will be constant and the oscillations will be in phase. The angular frequency will be ω_o.

The initial conditions for the odd mode are $\theta_1 = -\theta_2$, $\dot{\theta}_1 = -\dot{\theta}_2$ at t = 0. In this case, we have oscillations 180^o out of phase, and it can be shown that the frequency is given by

$$\omega^2 = \omega_0{}^2 + \frac{2k\ell^2}{mL^2} \tag{8a}$$

For small coupling, the binomial approximation yields

$$\omega = \omega_0 + \frac{k\ell^2}{\omega_0 mL^2} \tag{8b}$$

For any arbitrary initial conditions, the oscillation of each pendulum is a superposition of the normal modes. Of particular simplicity is the case where at time t = 0 we have $\theta_1 = \dot{\theta}_1 = 0$, and $\dot{\theta}_2 = 0$ but $\theta_2 = \theta_0$, some arbitrary value. Hence we are starting the pendulums with one having no energy, and the other having some energy, i.e. they begin swinging 90^o out of phase. For these initial conditions the energy is completely transferred back and forth between pendulums, with the frequency of the exchange determined by the degree of coupling. The superposition of the normal modes will give rise to beats, with the beat frequency (angular) given by

$$\Delta\omega = \frac{k\ell^2}{\omega_0 mL^2} \tag{9}$$

When other initial conditions are used, the beats are not complete, i.e. neither pendulum will ever be instantaneously at rest at the origin.

Experiment 5: Procedure and Analysis

(A) Determining the Spring Constant, $k \pm \Delta k$

Select 4 springs. Hang a mass from one spring and measure the associated displacement. Repeat for 6-8 masses and for each spring. In your report analyze your mass and displacement data to find $k \pm \Delta k$ for each spring.

(B) Normal Modes and Coupling Frequency

Record the mass, the four coupling lengths, and the total pendulum length for each of the pendulums. In your analysis you should use the average of each complementary value to determine theoretical values.

Measure the amplitude versus time signal for each pendulum. Data should be taken for each possible combination of spring constant, k, and mounting location, l, for both even and odd initial conditions (32 trials in total).

Note: Before taking data you should determine the best sampling frequency and total sampling time to ensure a good Fourier transform for your analysis. This means you should try to answer as much of questions 1 and 2 from Appendix A as you can **BEFORE** coming to the lab.

Use Fourier analysis to determine the angular frequency ω from each dataset (64 values). Provide an appropriate analysis of the frequencies in each case compared to theoretical predictions.

Hint: For the odd trials determine the slopes and intercepts of a plot of ω^2 vs. k and of ω^2 vs. l^2, with the associated error limits, using equation 8a and compare with calculated values. Note: All ω^2 vs k plots should be done on one graph, and all ω^2 vs l^2 on another.

(C) Beat Frequency

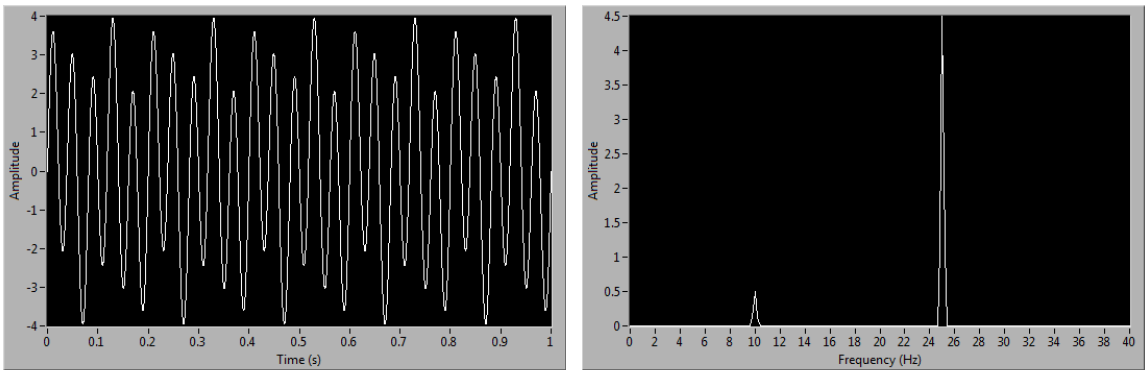
Measure the amplitude signal versus time for four beat trials. Each trial should use a different combination of spring constant, k, and mounting location, l. When choosing appropriate k and l values you should consider the small coupling approximation used to derive equations 8b and 9.

Using Fourier analysis techniques determine the beat frequencies for each trial and compare them to the theoretical values predicted by equation 9.

Appendix A: Some Notes on Fourier Analysis

You will need to obtain a copy of Igor Pro (or have access to other similar software, e.g. Matlab) in order to perform the Fourier analysis for this experiment.

Fourier analysis is a technique used to transform data taken in the time domain into data in the frequency domain. The Fourier transform indicates the strength of each component frequency in the time domain signal by returning the frequency strength as amplitude on a Fourier amplitude versus frequency plot. For example, if one were to take the FFT of the function $y = A\sin(\omega_1 t) + B\sin(\omega_2 t)$ where $A < B$ we would get,



Where the left plot is the time domain signal and the right is the Fourier transform. The FFT plot tells us that the wave was composed of two sine waves of frequencies 10Hz and 25Hz. The 25Hz component clearly had the stronger amplitude in the time domain signal.

In your report use the answers to the following questions to discuss your results:

1) Determine the separation in frequency between each point in the frequency domain spectrum. This can be used to place an uncertainty on your frequency values. What determines this separation in the original time domain data?

2) What is the maximum frequency you are able to observe in your Fourier transform? What property of the time domain data determines the maximum observable frequency?

Information on using the Fourier Transform analysis in Igor Pro can be found here:
http://www.wavemetrics.net/doc/igorman/III-09%20Signal%20Processing.pdf

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