Coupled Oscillators

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Section 1
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I. ABSTRACT

A wide variety of physical systems exhibit some form of oscillatory motion, such as the coupled vibrations between adjacent atoms and molecules. The importance of these oscillations to properties such as radiative interactions motivates the application of a mathematical framework to describe coupled oscillations and tools to simplify the extraction of characteristic frequencies from complicated waveforms. To this end, a simple form of oscillatory motion is studied in the case of two pendulums coupled together by a spring. The method of normal coordinates is applied to obtain theoretical values for the oscillatory frequencies. These values are compared to those obtained from the Fourier transforms of data sets of odd, even, and beat oscillations of the pendulums

II. INTRODUCTION

Oscillatory motion can be found in most physical phenomena, including electronic circuits, n-body systems, and the quantum harmonic potential. This allows for a wide variety of complicated systems to be modelled using comparatively straightforward mathematical methods. Crystal lattices and atomic nuclei are one such system, as the interactions between neighbouring atoms manifest themselves in vibrational modes and affect a variety of atomic properties. These motions allow for specific frequencies that depend on atomic properties such as mass and charge, and on the nature of the specific interactions involved. It is understood that these coupled oscillations significantly affect how atomic or molecular structures interact with radiation, so there exists a clear motivation to find a reliable mathematical framework to model them.

The Fourier transform is a useful mathematical tool for analyzing systems exhibiting periodic motion. It transforms the domain of an oscillatory input from time to frequency, allowing for the simple identification of oscillatory frequencies. This has substantial applications to the previously mentioned physical systems. In our experiment, two pendulums are coupled together with springs of varying spring constants, and at different coupling heights, to provide a simple model of adjacent atoms in a lattice. The pendulums are allowed to oscillate in both their even and odd modes, and the resulting waveforms are Fourier transformed to obtain their frequency spectra, so that they may be compared to the theoretically predicted results.

III. THEORETICAL BACKGROUND

When objects such as atoms are placed in spaces where they can interact with one another, they undergo oscillatory motion due to the forces they exert on one another, and on the quantization of their available energy states. To understand the physical properties of such systems it is important to characterize this oscillatory behaviour. Radiation, for example, by either adding or subtracting energy from the system, directly affects the allowable frequencies of vibration that the atoms can undergo (not to mention that photons themselves are oscillating electro-magnetic waves). To model such systems, a simple model comprising two pendulums can be constructed. The springs coupling the two pendulums, with their respective spring constants and mounting locations along the length of the pendulums, can be thought to represent the strength of the interaction between two adjacent atoms. For a simple pendulum of length L, the restoring torque is given by:

$$\Gamma = -I\ddot{\theta} \tag{1}$$

Where θ is the angle between the pendulum and its equilibrium position (with θ being the corresponding acceleration), and the moment of inertia, I, given by mL^2, where m is the mass at the end of the pendulum (the pendulum itself is assumed to be a massless string). In this case, the restoring force is gravity, so the equation can be re-written as:

$$mL^2\ddot{\theta} = -mgL\sin\theta \tag{2}$$

Applying the small angle approximation $\sin\theta \approx \theta$ gives the well known equation $\ddot{\theta} = -\frac{g}{L}\theta$ with solutions $\theta = \theta_0 \cos\omega_0 t$. Substituting the solutions in produces a formula for the angular frequency of a single pendulum,

$$\omega_0 = \sqrt{\frac{g}{L}} \tag{3}$$

Extending this approach to two pendulums coupled together necessitates the addition of a spring term to Eq.2, which depend on the relative angle between the pendulums, and on the distance, l, between from the pivot to the spring mounting. Now, with indices denoting the form for each pendulum,

$$mL^2\ddot{\theta}_{1,2} = -mgL\sin\theta_{1,2} + k\ell^2(\sin\theta_{2,1} - \sin\theta_{1,2}) \tag{4}$$

Applying the small angle approximation once again gives:

$$\ddot{\theta}_{1,2} = -\frac{g}{L}\theta_{1,2} + \frac{k\ell^2}{mL^2}(\theta_{2,1} - \theta_{1,2}) \tag{5}$$

It is possible to select, for this set of equations, a pair of co-ordinates or nodes, relative to which all parts of the system oscillate with the same phase and frequency. These are called normal modes. There are a total of two for the coupled pendulums, the even mode and the odd mode, which correspond to symmetrical and anti-symmetrical solutions respectively. The former is defined by $\theta_1 = \theta_2$ and $\dot{\theta}_1 = \dot{\theta}_2$, while the latter has $\theta_1 = -\theta_2$ and $\dot{\theta}_1 = -\dot{\theta}_2$ at the initial time.

In the case of the even mode, the pendulums will oscillate as if they were uncoupled, as they share a common phase and will not experience restoring forces from the spring. As for a single pendulum, the frequency of oscillations is given by Eq.3. On the other hand, the odd mode introduces a phase difference of π, so that $\theta_1 = -\theta_2$.

Since the motion is expected to be oscillatory, the following change of variables is made:

$$\theta_1 = A_1 e^{i\omega t} = -A_2 e^{i\omega t} = -\theta_2 \tag{6}$$

Equation 5 can now be rewritten as $-\omega^2 e^{i\omega t} = -\frac{g}{L}e^{i\omega t} + \frac{k\ell^2}{mL^2}\cdot(-2e^{i\omega t})$, which simplifies further to:

$$\omega^2 = \omega_0^2 + \frac{2k\ell^2}{mL^2} \tag{7}$$

and

$$\omega = \omega_0 + \frac{k\ell^2}{\omega_0 mL^2} \tag{8}$$

This last equation is obtained by applying the binomial approximation to Eq.7.

The two normal mode coordinates used in this method form a basis consisting of the even and odd modes. It can therefore be inferred that oscillations which do not match the initial conditions of either normal coordinate must be super-positions of the two. Initial conditions that fall within this category give rise to beats in the oscillations. Of particular interest are initial conditions where one pendulum is at rest at its equilibrium position ($\theta_1 = \dot{\theta}_1 = 0$), while the other is at rest ($\dot{\theta}_2 = 0$) at some arbitrary value of θ_2. This creates a phase difference of $\frac{\pi}{2}$ between the oscillations of the two pendulums, and the results in the energy passing completely from one pendulum to the other (and back again).

It should be noted that the frequency of oscillations of the even and odd modes differ by the term $\frac{k\ell^2}{\omega_0 mL^2}$. This is the frequency of the beats. This summarizes the theoretical description of the behaviour of coupled pendulums. All that is needed from this point is a set of mathematical tools to break up the experimental data obtained from their oscillations to extract the frequencies and compare them to the theory. The Fourier transform is just such a tool.

The Fourier transform is motivated by the applications of Fourier series, and by the representation of periodic functions as the sums of sine and cosine waves. From the theory of Fourier series, it is known that a periodic function, f(x), with period T, can be broken down into a sum of exponentials of the form $\sum_{-N}^{N} c_n e^{i\frac{2\pi nx}{T}}$, with N being an integer greater than or equal to 1, and where the nth coefficients are given by $c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(x)e^{i\frac{2\pi nx}{T}}dx$. For a continuous function, this allows for the definition of the Fourier transform, $F(\omega)$, where $F(\omega) = \int_{-\infty}^{\infty} f(x)e^{i\omega x}dx$, with ω becoming the angular frequency by the change of variable $\omega = \frac{2\pi n}{T}$. This lends itself well to the analysis of the signals obtained from the motions of coupled pendulums. By decomposing the complicated waveforms into their much simpler frequency spectra, it becomes much simpler to characterize the oscillations with their normal modes.

IV. EXPERIMENTAL DESIGN AND PROCEDURE

1. Four springs with different spring constants were selected to be used in the experiment. Six (known) masses were hung from each one, and the associated spring lengths were recorded.
2. Using the displacements for each spring, their respective spring constants were calculated.
3. One of the springs was attached to the pendulum set up. This set up consisted of two pendulums, attached a bar of fixed height, with spring attachment points at identical distances along the length of each pendulum. The set up was hooked up to a computer running Igor Pro, a software suite which would record the motions of each pendulum along the x-y plane.
4. The masses attached to the pendulums were recorded.
5. The masses were allowed to oscillate in the even, and then odd modes for about a dozen cycles, with data being recorded in Igor Pro.
6. The spring was removed and step 5 was repeated for three more mounting locations (identical ones simultaneously on both pendulums).
7. Steps 5 and 6 were repeated for the 3 other springs.
8. After attaching one of the springs to the pendulums, only one pendulum was raised up and let go to create the $\frac{\pi}{2}$ phase shift necessary for the beat frequencies. The results were again recorded in Igor Pro.
9. Step 8 was repeated for the 3 other mounting locations, and again at each location for a second spring.

V. ANALYSIS

Recorded Masses of Pendulums:
$m_1 = 2931$ g
$m_2 = 2946$ g

Sample Calculations for $\overline{m}$

$$\overline{m} = \frac{\sum_i^n d_i}{n}$$
$$\overline{m} = \frac{m_1 + m_2}{2}$$
$$\overline{m} = \frac{2931g + 2946g}{2}$$
$$\overline{m} = 2939g$$

Mounting Location	Length from Pivot (±0.1cm)	Average Length (±0.1cm)
1	13.4, 13.5	13.4
2	22.2, 22.1	22.2
3	30.5, 30.2	30.4
4	41.6, 41.6	41.6
(to mass)	55.9, 56.1	56.0

TABLE I: Measurements of the mounting lengths and the distance to the center of mass from the pivot point of each pendulum.

Sample Calculations for Average Length using row 1 of Table I

$$\overline{d} = \frac{\sum_i^n d_i}{n}$$
$$\overline{d} = \frac{13.4cm + 13.5cm}{2}$$
$$\overline{d} = 13.4cm$$

Sample Calculations for Δ Average Length using row 1 of Table I

$$\Delta\overline{d} = \sqrt{\Delta d^2 \times 2}$$
$$\Delta\overline{d} = \sqrt{0.1cm^2 \times 2}$$
$$\Delta\overline{d} = 0.14cm$$
$$\Delta\overline{d} \approx 0.1cm$$

A. Determination of Spring Constants

Spring 1		Spring 2		Spring 3		Spring 4	
Mass (g)	Displacement (± 0.05 cm)	Mass (g)	Displacement (± 0.05 cm)	Mass (g)	Displacement (± 0.05 cm)	Mass (g)	Displacement (± 0.05 cm)
50	27.6	50	34.3	50	31.6	50	34.9
70	30.8	70	35.0	100	31.9	90	35.7
90	34.1	90	35.6	150	32.2	130	36.4
110	37.3	110	36.2	200	32.5	180	37.3
130	40.6	130	36.8	280	33.0	1 220	38.1
–	–	150	37.5	550	34.6	280	39.1

TABLE II: Masses applied and associated displacements to Springs 1-4

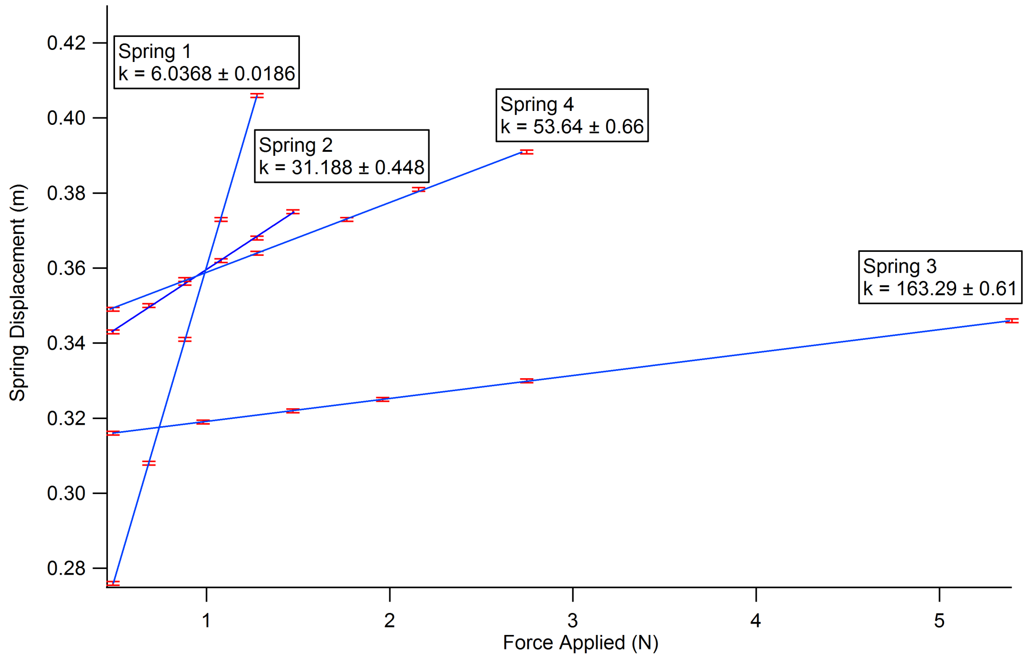

FIG. 1: Spring Displacement versus Force Applied using masses, for Springs 1-4.

The spring constants for the 4 springs were determined by graphing the associated Displacement versus Force Applied, the spring constants from least to most rigid was $k_1 = 6.0368 \pm 0.0186\ \frac{N}{m}$, $k_2 = 31.188 \pm 0.448\ \frac{N}{m}$, $k_4 = 53.64 \pm 0.66\ \frac{N}{m}$, and last $k_3 = 163.29 \pm 0.61\ \frac{N}{m}$.

B. Normal Modes and Coupling Frequency

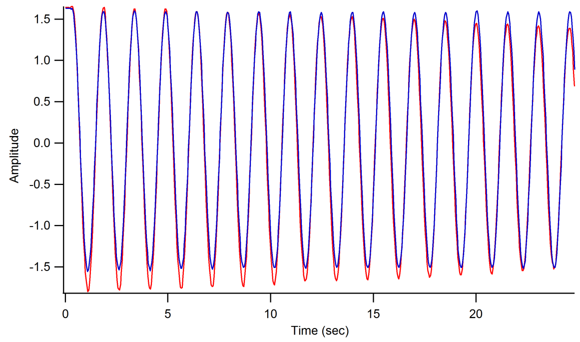

FIG. 2: Periodic Motion of the pendulums in an even mode initial configuration. Spring 1 is mounted onto the 2nd mounting position (ℓ_2).

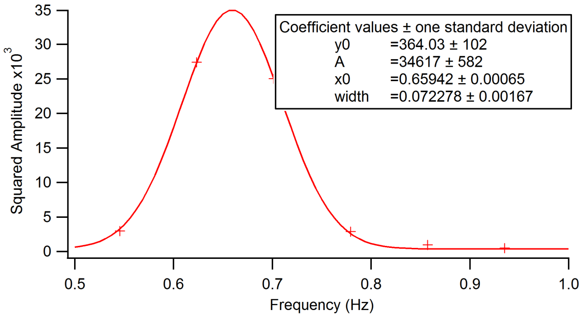

FIG. 3: Fourier Analysis of Periodic Motion of the pendulums (as seen in Fig.4) in an even mode initial configuration. Spring 1 is mounted onto the 2nd mounting position (ℓ_2).

Spring No.	ℓ	ν_1 (Hz)	$\Delta\nu_1$ (Hz)	ω_1 $(\frac{rad}{sec})$	$\Delta\omega_1$ $(\frac{rad}{sec})$	ν_2 (Hz)	$\Delta\nu_2$ (Hz)	ω_2 $(\frac{rad}{sec})$	$\Delta\omega_2$ $(\frac{rad}{sec})$	$\overline{\omega}$ $(\frac{rad}{sec})$	$\Delta\overline{\omega}$ $(\frac{rad}{sec})$	% Error
1	1	0.662870	0.000571	4.164935	0.003588	0.661430	0.000618	4.155887	0.003883	4.160411	0.005287	0.7
	2	0.659420	0.000650	4.143258	0.004084	0.655640	0.000776	4.119508	0.004876	4.131383	0.006360	1.4
	3	0.656870	0.000687	4.127236	0.004317	0.657080	0.000680	4.128555	0.004273	4.127896	0.006074	1.5
	4	0.663890	0.000603	4.171344	0.003789	0.665940	0.000627	4.184224	0.003940	4.177784	0.005466	0.3
2	1	0.654530	0.000821	4.112533	0.005158	0.657440	0.000716	4.130817	0.004499	4.121675	0.006845	1.6
	2	0.661180	0.000635	4.154316	0.003990	0.661730	0.000596	4.157772	0.003745	4.156044	0.005472	0.8
	3	0.660970	0.000641	4.152997	0.004028	0.659750	0.000717	4.145332	0.004505	4.149164	0.006043	1.0
	4	0.663770	0.000706	4.170590	0.004436	0.664580	0.000502	4.175679	0.003154	4.173135	0.005443	0.4
3	1	0.673690	0.000987	4.232919	0.006202	0.672950	0.000876	4.228270	0.005504	4.230594	0.008292	1.0
	2	0.675000	0.000860	4.241150	0.005404	0.676080	0.000975	4.247936	0.006126	4.244543	0.008169	1.3
	3	0.668340	0.000694	4.199304	0.004361	0.668970	0.000685	4.203262	0.004304	4.201283	0.006127	0.3
	4	0.666700	0.000744	4.189000	0.004675	0.666770	0.000792	4.189439	0.004976	4.189220	0.006828	0.0
4	1	0.665650	0.000912	4.182402	0.005730	0.660370	0.000757	4.149227	0.004756	4.165815	0.007447	0.6
	2	0.661250	0.000684	4.154756	0.004298	0.660430	0.000716	4.149604	0.004499	4.152180	0.006222	0.9
	3	0.670570	0.000582	4.213316	0.003657	0.669330	0.000732	4.205524	0.004599	4.209420	0.005876	0.5
	4	0.660180	0.000725	4.148033	0.004555	0.660000	0.000744	4.146902	0.004675	4.147468	0.006527	1.0

TABLE III: Frequency values from Fourier Analysis of amplitude data for Pendulum 1 & 2 for Even modes.

From the even mode trials, where the pendulums are effectively uncoupled, the angular frequency of the pendulums, ω_0 was measured and found using a Fourier transform on the date collected. Using a Gaussian regression on the Fourier Transform data, the peak frequency was able to be determined and all the values averaged over the 16 trials and was found to be $4.171126 \pm 0.023111\ \frac{rad}{sec}$. The calculated error ranged from 0-1.5% of the theoretically calculated value of $4.19\ \frac{rad}{sec}$.

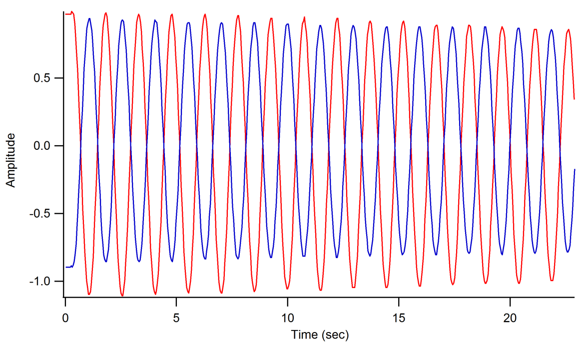

FIG. 4: Periodic Motion of the pendulums in an odd mode initial configuration. Spring 1 is mounted onto the 2nd mounting position (ℓ_2).

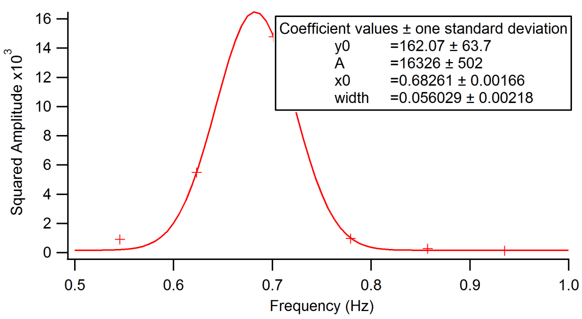

FIG. 5: Fourier Analysis of Periodic Motion of the pendulums (as seen in Fig.4) in an odd mode initial configuration. Spring 1 is mounted onto the 2nd mounting position (ℓ_2).

Spring No.	ℓ	ν_1 (Hz)	$\Delta\nu_1$ (Hz)	ω_1 $(\frac{rad}{sec})$	$\Delta\omega_1$ $(\frac{rad}{sec})$	ν_2 (Hz)	$\Delta\nu_2$ (Hz)	ω_2 $(\frac{rad}{sec})$	$\Delta\omega_2$ $(\frac{rad}{sec})$	$\overline{\omega}$ $(\frac{rad}{sec})$	$\Delta\overline{\omega}$ $(\frac{rad}{sec})$
1	1	0.673540	0.000990	4.231977	0.006220	0.673560	0.000927	4.232102	0.005825	4.232039	0.008522
	1	0.682610	0.001660	4.288965	0.010430	0.684070	0.001700	4.298139	0.010681	4.293552	0.014929
	1	0.693500	0.001160	4.357389	0.007288	0.694430	0.001120	4.363232	0.007037	4.360311	0.010131
	1	0.699550	0.000239	4.395402	0.001502	0.698870	0.000215	4.391130	0.001351	4.393266	0.002020
2	1	0.694050	0.000999	4.360845	0.006277	0.694930	0.000910	4.366374	0.005718	4.363609	0.008491
	2	0.711820	0.002130	4.472497	0.013383	0.701240	0.000814	4.406021	0.005115	4.439259	0.014327
	3	0.763290	0.003880	4.795893	0.024379	0.770230	0.003760	4.839498	0.023625	4.817695	0.033948
	4	0.791650	0.001230	4.974084	0.007728	0.791090	0.001160	4.970565	0.007288	4.972324	0.010623
3	1	0.771660	0.000668	4.848483	0.004197	0.774670	0.001180	4.867395	0.007414	4.857939	0.008520
	2	0.880020	0.001590	5.529329	0.009990	0.877880	0.001560	5.515883	0.009802	5.522606	0.013996
	3	1.022000	0.001430	6.421415	0.008985	1.023100	0.001270	6.428327	0.007980	6.424871	0.012017
	4	1.318600	0.002770	8.285008	0.017404	1.321900	0.002250	8.305743	0.014137	8.295375	0.022423
4	1	0.703760	0.000541	4.421854	0.003399	0.699530	0.000408	4.395277	0.002564	4.408566	0.004258
	2	0.760330	0.001170	4.777294	0.007351	0.764060	0.002090	4.800731	0.013132	4.789012	0.015050
	3	0.809020	0.001130	5.083223	0.007100	0.806960	0.000954	5.070279	0.005994	5.076751	0.009292
	4	0.928300	0.001230	5.832681	0.007728	0.928270	0.001310	5.832492	0.008231	5.832587	0.011291

TABLE IV: Frequency values from Fourier Analysis of amplitude data for Pendulum 1 & 2 for Odd modes.

From the odd mode trials, where the pendulums coupled, the angular frequency of the pendulums, ω was measured and found using a Fourier transform on the date collected. Using a gaussian regression on the Fourier Transform data, the peak frequency was able to be determined, then $\omega^2 vsk$ was plotted in Fig.6, keeping the mounting location ℓ constant for the 4 data sets plotted, as well as $\omega^2 vs\ell^2$ in Fig.7 plotted keeping the spring constant k constant for the 4 data sets plotted. The lines of best fit were found, and the slopes were compared to the theoretical values.

Sample Calculations for ω_1 using row 1 of Table III

$$\nu = \frac{1}{2\pi}\omega$$
$$\omega_1 = 2\pi \times \nu_1$$
$$\omega_1 = 2\pi \times 0.662870$$
$$\omega_1 = 4.164935$$

Sample Calculations for $\Delta\omega_1$ using row 1 of Table III

$$\Delta\omega_1 = 2\pi \times \Delta\nu_1$$
$$\Delta\omega_1 = 2\pi \times 0.000571$$
$$\Delta\omega_1 = 0.003588$$

Sample Calculations for $\overline{\omega}$ using row 1 of Table III

$$\overline{\omega} = \frac{\sum_i^n d_i}{n}$$
$$\overline{\omega} = \frac{4.164935+4.155887}{2}$$
$$\overline{\omega} = 4.160411$$

Sample Calculations for $\Delta\overline{\omega}$ using row 1 of Table III

$$\Delta\overline{d} = \sqrt{\Delta\omega_1^2 + \Delta\omega_2^2}$$
$$\Delta\overline{d} = \sqrt{0.003588^2 + 0.003883^2}$$
$$\Delta\overline{d} = 0.005287$$

Sample Calculations for Theoretical ω_0 using row 1 of Table III, using g = 9.81 $\frac{m}{s^2}$, L = 0.56 m

$$\omega_0 = \sqrt{\frac{g}{L}}$$
$$\omega_0 = \sqrt{\frac{9.81}{0.56}}$$
$$\omega_0 = 4.19$$

Sample Calculations for Percent Error using row 1 of Table III

$$\%Error = \frac{|Theoretical - Experimental|}{Theoretical} \times 100\%$$
$$\%Error = \frac{|\omega_0 - \omega|}{\omega_0} \times 100\%$$
$$\%Error = \frac{|4.19 - 4.160411|}{4.19} \times 100\%$$
$$\%Error = 0.706\%$$
$$\%Error \approx 0.7\%$$

Calculations for ω_0 using Table III

$$\omega_0 = \frac{\sum_i^n d_i}{n}$$
$$\omega_0 = \frac{\begin{array}{c}4.160411 + 4.131383 + 4.127896 + 4.177784 + 4.121675 + 4.156044 + 4.149164 + 4.173135 + \\ 4.230594 + 4.244543 + 4.201283 + 4.189220 + 4.165815 + 4.152180 + 4.209420 + 4.147468\end{array}}{16}$$
$$\omega_0 = 4.171126$$

Calculations for $\Delta\omega_0$ using Table III

$$\Delta\omega_0 = \sqrt{\sum_i^n \Delta\omega_i^2}$$
$$\Delta\omega_0 = \sqrt{\begin{array}{c}0.005287^2 + 0.006360^2 + 0.006074^2 + 0.005466^2 + 0.006845^2 + 0.005472^2 + 0.006043^2 + 0.005443^2 + \\ 0.008292^2 + 0.008169^2 + 0.006127^2 + 0.006828^2 + 0.007447^2 + 0.006222^2 + 0.005876^2 + 0.006527^2\end{array}}$$
$$\Delta\omega_0 = 0.023111$$

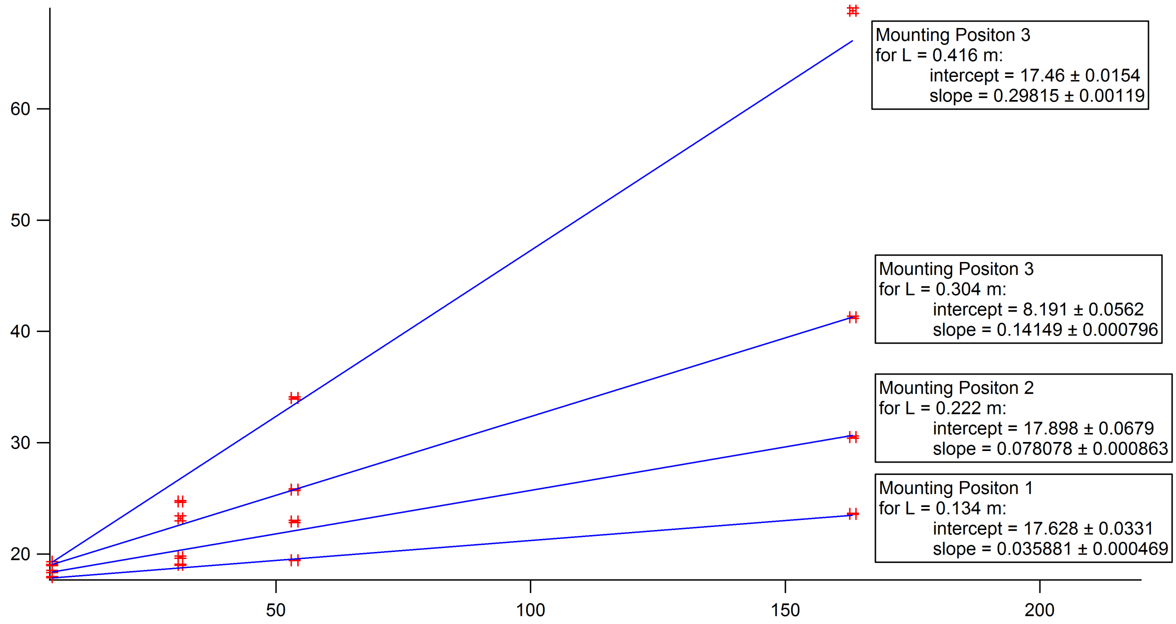

FIG. 6: Angular frequency of Odd modes squared (ω^2) versus Spring Constant (k) while keeping mounting position (ℓ) constant over all 4 mounting positions.

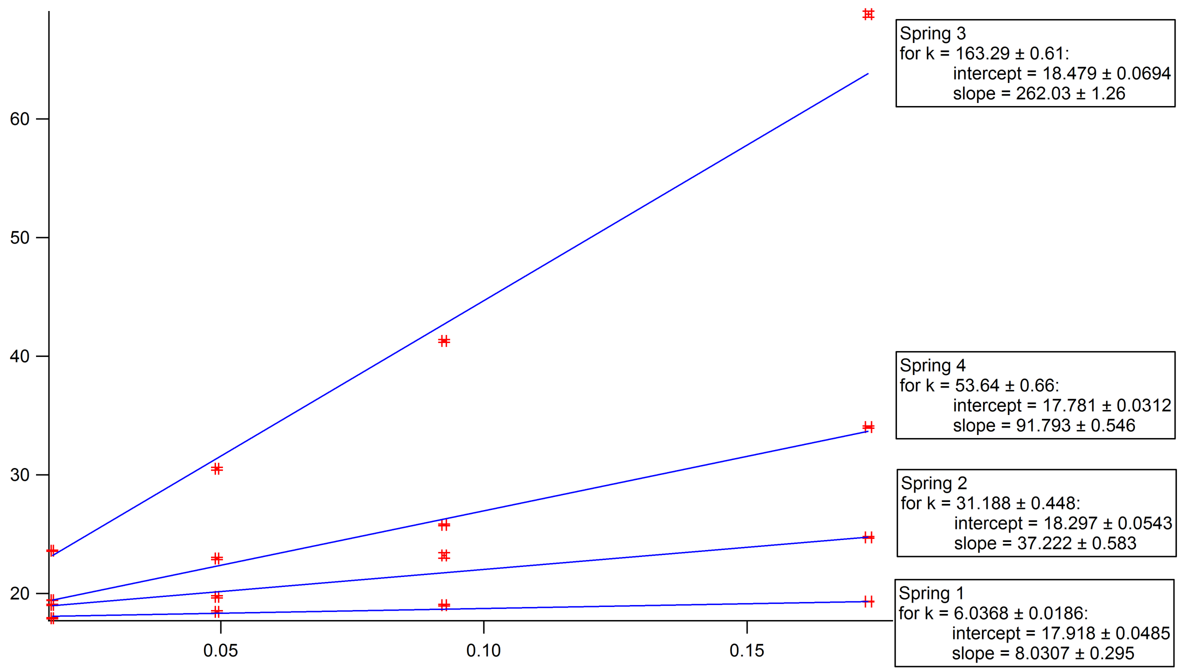

FIG. 7: Angular frequency of Odd modes squared (ω^2) versus Mounting Position squared (ℓ^2) while keeping the Spring Constant (k) constant for all 4 springs.

ℓ (m)	$\frac{\omega^2}{k}$ (kg^{-1})	$\Delta\frac{\omega^2}{k}$ (kg^{-1})	Theoretical $\frac{\omega^2}{k}$ (kg^{-1})	Δ Theoretical $\frac{\omega^2}{k}$ (kg^{-1})	% Error
0.134	0.035881	0.000469	3.90 $\times 10^{-2}$	1.09 $\times 10^{-2}$	8.0
0.222	0.078078	0.000863	1.07 $\times 10^{-1}$	6.85 $\times 10^{-3}$	27.0
0.304	0.14149	0.000796	2.01 $\times 10^{-1}$	5.29 $\times 10^{-3}$	29.6
0.416	0.29815	0.000119	3.76 $\times 10^{-1}$	4.23 $\times 10^{-3}$	20.7

TABLE V: Tabulated Results from Fig.6, ω^2 versus k, and calculated theoretical values for the modes for springs mounted at the various mounting levels and associated error values.

Sample Calculations for Theoretical $\frac{\omega^2}{k}$ using row 1 of Table V, using m = 2.939 kg, L = 0.56 m

$$\omega^2 = \omega_0^2 + \frac{2 \cdot k\ell^2}{mL^2}$$
$$slope = \frac{\omega^2}{k} = \frac{2 \cdot \ell^2}{mL^2}$$
$$\frac{\omega^2}{k} = \frac{2 \cdot 0.134^2}{2.939 \times 0.56^2}$$
$$\frac{\omega^2}{k} = 0.03896$$
$$\frac{\omega^2}{k} \approx 3.90 \times 10^{-2} kg^{-1}$$

Sample Calculations for Δ Theoretical $\frac{\omega^2}{k}$ using row 1 of Table V, using L = 0.56 m

$$\Delta Theoretical \frac{\omega^2}{k} = \sqrt{(\frac{\Delta\ell}{\ell})^2 \times 2 + (\frac{\Delta L}{L})^2 \times 2}$$
$$\Delta Theoretical \frac{\omega^2}{k} = \sqrt{(\frac{0.001}{0.134})^2 \times 2 + (\frac{0.001}{0.56})^2 \times 2}$$
$$\Delta Theoretical \frac{\omega^2}{k} = 0.01085$$
$$\Delta Theoretical \frac{\omega^2}{k} \approx 1.09 \times 10^{-2} kg^{-1}$$

Sample Calculations for Percent Error using row 1 of Table V

$$\%Error = \frac{|Theoretical - Experimental|}{Theoretical} \times 100\%$$
$$\%Error = \frac{|3.90 \times 10^{-2} - 0.035881|}{3.90 \times 10^{-2}} \times 100\%$$
$$\%Error = 7.997\%$$
$$\%Error \approx 8.0\%$$

k (m)	$\frac{\omega^2}{\ell^2}$ $(\frac{N}{kg \cdot m^3})$	$\Delta\frac{\omega^2}{\ell^2}$ $(\frac{N}{kg \cdot m^3})$	Theoretical Value $(\frac{N}{kg \cdot m^3})$	Δ Theoretical Value $(\frac{N}{kg \cdot m^3})$	% Error
6.0368	8.0307	0.295	1.31 $\times 10^{1}$	3.98 $\times 10^{-3}$	38.7
31.188	37.222	0.583	6.77 $\times 10^{1}$	1.58 $\times 10^{-2}$	45.0
53.64	91.793	0.546	1.16 $\times 10^{2}$	1.26 $\times 10^{-2}$	20.9
163.29	262.03	1.26	3.54 $\times 10^{2}$	4.51 $\times 10^{-3}$	26.0

TABLE VI: Tabulated Results from Fig.7, ω^2 versus ℓ^2, and calculated theoretical values for the modes for the various spring constants and associated error values.

From the graphs of $\omega^2 vs k$ in Fig.6 and $\omega^2 vs \ell^2$ in Fig.7, comparing to theoretical values there was a significant amount of error, ranging from about 8.0% to as much as 45%. During the experiment, there was some unstable oscillations transverse to the normal motion of the pendulum, due to the stiffness of the spring and unstable mounting (the spring just hooked onto the mounting locations rather than being fixed strongly to the bracket) allowing for bumping and slipping actions during the oscillations. Doing more trials with more spring may have been able to help with the regression to give a more accurate set of points for the line of best fit.

Sample Calculations for Theoretical $\frac{\omega^2}{\ell^2}$ using row 1 of Table VI, using m = 2.939 kg, L = 0.56 m

$$\omega^2 = \omega_0^2 + \frac{2 \cdot k \ell^2}{mL^2}$$
$$slope = \frac{\omega^2}{\ell^2} = \frac{2 \cdot k}{mL^2}$$
$$\frac{\omega^2}{\ell^2} = \frac{2 \cdot 6.0368}{2.939 \times 0.56^2}$$
$$\frac{\omega^2}{\ell^2} = 13.0997$$
$$\frac{\omega^2}{\ell^2} \approx 1.31 \times 10^1 \frac{N}{kg \cdot m^3}$$

Sample Calculations for Δ Theoretical $\frac{\omega^2}{\ell^2}$ using row 1 of Table VI, using Δk_1 = 0.0186 $\frac{N}{m}$

$$\Delta Theoretical \frac{\omega^2}{\ell^2} = \sqrt{(\frac{\Delta k}{k})^2 + (\frac{\Delta L}{L})^2 \times 2}$$
$$\Delta Theoretical \frac{\omega^2}{\ell^2} = \sqrt{(\frac{0.0186}{6.0368})^2 + (\frac{0.001}{0.56})^2 \times 2}$$
$$\Delta Theoretical \frac{\omega^2}{\ell^2} = 3.98 \times 10^{-3} \frac{N}{kg \cdot m^3}$$

Sample Calculations for Percent Error using row 1 of Table VI

$$\%Error = \frac{|Theoretical - Experimental|}{Theoretical} \times 100\%$$
$$\%Error = \frac{|13.1 - 8.0307|}{13.1} \times 100\%$$
$$\%Error = 38.697\%$$
$$\%Error \approx 38.7\%$$

C. Beat Frequency

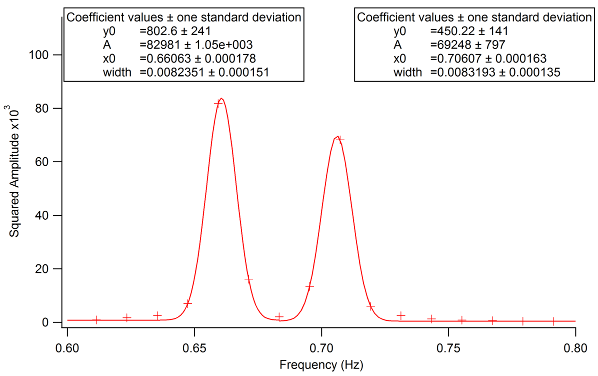

FIG. 8: Fourier Analysis of beat frequency on Pendulum 1 with Spring 1 (k = 6.0368 $\frac{N}{m}$) in mounting position 4 (ℓ = 0.416).

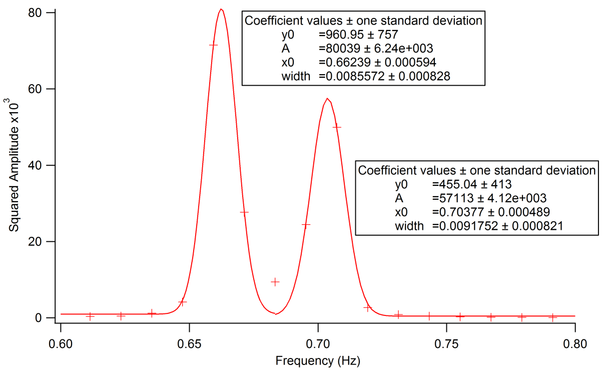

FIG. 9: Fourier Analysis of beat frequency on Pendulum 2 with Spring 1 (k = 6.0368 $\frac{N}{m}$) in mounting position 4 (ℓ = 0.416).

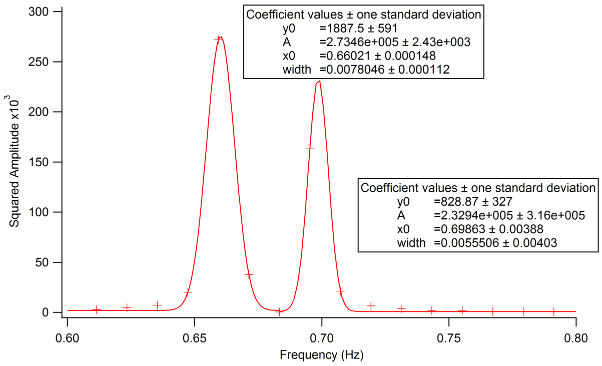

FIG. 10: Fourier Analysis of beat frequency on Pendulum 1 with Spring 4 ($k = 53.64\ \frac{N}{m}$) in mounting position 1 ($\ell = 0.134$).

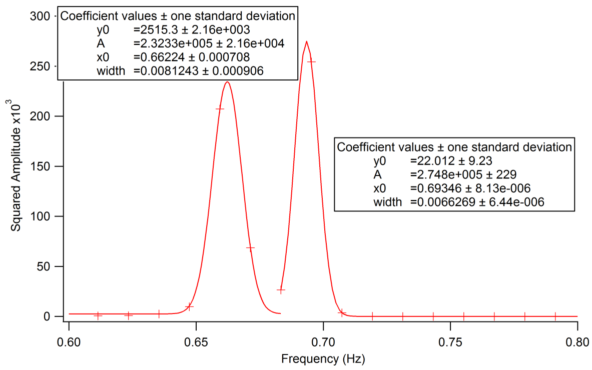

FIG. 11: Fourier Analysis of beat frequency on Pendulum 2 with Spring 4 ($k = 53.64\ \frac{N}{m}$) in mounting position 1 ($\ell = 0.134$).

k ($\frac{N}{m}$)	ℓ (m)	ν_{1a} (Hz)	$\Delta\nu_{1a}$ (Hz)	ν_{2a} (Hz)	$\Delta\nu_{2a}$ (Hz)	ν_{1b} (Hz)	$\Delta\nu_{1b}$ (Hz)	ν_{2b} (Hz)	$\Delta\nu_{2b}$ (Hz)
6.0368	0.416	0.66063	0.000178	0.70607	0.000163	0.66239	0.000594	0.70377	0.000489
53.64	0.134	0.66021	0.000148	0.69863	0.00388	0.66224	0.000708	0.69346	0.000008

TABLE VII: Tabulated Results from Fourier Analysis of Beat Frequencies within the Couples Oscillator system for the two springs in the two set mounting positions.

ω_{1a} ($\frac{rad}{sec}$)	$\Delta\omega_{1a}$ ($\frac{rad}{sec}$)	ω_{2a} ($\frac{rad}{sec}$)	$\Delta\omega_{2a}$ ($\frac{rad}{sec}$)	ω_{1b} ($\frac{rad}{sec}$)	$\Delta\omega_{1b}$ ($\frac{rad}{sec}$)	ω_{2b} ($\frac{rad}{sec}$)	$\Delta\omega_{2b}$ ($\frac{rad}{sec}$)	$\overline{\omega_1}$ ($\frac{rad}{sec}$)	$\Delta\overline{\omega_1}$ ($\frac{rad}{sec}$)	$\overline{\omega_2}$ ($\frac{rad}{sec}$)	$\Delta\overline{\omega_2}$ ($\frac{rad}{sec}$)
4.15086	0.00112	4.43637	0.00102	4.16192	0.00373	4.42192	0.00307	4.15639	0.00390	4.42914	0.00324
4.14822	0.00093	4.38962	0.02438	4.16098	0.00445	4.35714	0.00005	4.15460	0.00454	4.37338	0.02438

TABLE VIII: Tabulated Results from Fourier Analysis of Beat Frequencies within the Couples Oscillator system for the two springs in the two set mounting positions.

Sample Calculations for ω_{1a} in row 1 of Table VIII, using row 1 of Table VII

$$\nu = \frac{1}{2\pi}\omega$$
$$\omega_{1a} = 2\pi \times \nu_{1a}$$
$$\omega_{1a} = 2\pi \times 0.66063$$
$$\omega_{1a} = 4.15086$$

Sample Calculations for $\Delta\omega_{1a}$ in row 1 of Table VIII, using row 1 of Table VII

$$\Delta\omega_{1a} = 2\pi \times \Delta\nu_{1a}$$
$$\Delta\omega_{1a} = 2\pi \times 0.000178$$
$$\Delta\omega_{1a} = 0.00112$$

Sample Calculations for $\overline{\omega_1}$ using row 1 of Table VIII

$$\overline{\omega_1} = \frac{\sum_i^n d_i}{n}$$
$$\overline{\omega_1} = \frac{\omega_{1a}+\omega_{1b}}{2}$$
$$\overline{\omega_1} = \frac{4.15086+4.16192}{2}$$
$$\overline{\omega_1} = 4.15639$$

Sample Calculations for $\Delta\overline{\omega_1}$ using row 1 of Table VIII

$$\Delta\overline{\omega_1} = \sqrt{\Delta\omega_{1a}^2 + \Delta\omega_{1b}^2}$$
$$\Delta\overline{\omega_1} = \sqrt{0.00112^2 + 0.00373^2}$$
$$\Delta\overline{\omega_1} = 0.00390$$

k ($\frac{N}{m}$)	ℓ (m)	$\omega_1 - \omega_2$ ($\frac{rad}{sec}$)	Theoretical $\omega_1 - \omega_2$ ($\frac{rad}{sec}$)	Theoretical $\Delta(\omega_1 - \omega_2)$ ($\frac{rad}{sec}$)	% Error
6.0368	0.416	0.27275	0.272	0.00762	0.3
53.6400	0.134	0.21878	0.251	0.01732	12.8

TABLE IX: The difference in dominant Frequencies, the Beat Frequency, and comparison to their Theoretical Values.

By using Fourier Analysis on the trials to find the beat frequency of the system, we were able to calculate, for two separate trials using a weak spring and a medium strength spring. Using by calculating theoretical values using $\Delta\omega = \omega_1 - \omega_2$ as the second term in Eq.8, the % Error was quite small (0.3%) for the trial with the weaker spring, but was larger with the medium strength spring (12.8%).

Sample Calculations for Theoretical $\omega_1 - \omega_2$ using row 1 of Table IX, using $\omega_0 = 4.171126\ \frac{rad}{sec}$, $m = 2.939$ kg, $L = 0.56$ m

$$\omega_1 - \omega_2 = \frac{k\ell^2}{\omega_0 m L^2}$$
$$\omega_1 - \omega_2 = \frac{6.0368 \cdot 0.416^2}{4.171126 \cdot 2.939 \cdot 0.56^2}$$
$$\omega_1 - \omega_2 = 0.272$$

Sample Calculations for Theoretical $\Delta(\omega_1 - \omega_2)$ using row 1 of Table IX, using $\omega_0 = 4.171126\ \frac{rad}{sec}$, $m = 2.939$ kg, $L = 0.56$ m

$$\Delta(\omega_1 - \omega_2) = \sqrt{(\frac{\Delta k}{k})^2 + (\frac{\Delta \ell}{ell})^2 \times 2 + (\frac{\Delta\omega_0}{\omega_0})^2 + (\frac{\Delta L}{L})^2 \times 2}$$
$$\Delta(\omega_1 - \omega_2) = \sqrt{(\frac{0.0186}{6.0368})^2 + (\frac{0.001}{0.416})^2 \times 2 + (\frac{0.023111}{4.171126})^2 + (\frac{0.001}{0.56})^2 \times 2}$$
$$\Delta(\omega_1 - \omega_2) = 0.00762\frac{rad}{sec}$$

Sample Calculations for Percent Error using row 1 of Table IX

$$\%Error = \frac{|Theoretical - Experimental|}{Theoretical} \times 100\%$$
$$\%Error = \frac{|0.272 - 0.27275|}{0.272} \times 100\%$$
$$\%Error = 0.2757\%$$
$$\%Error \approx 0.3\%$$

VI. CONCLUSION

The spring constants for the 4 springs were determined by graphing the associated Displacement versus Force Applied, the spring constants from least to most rigid was $k_1 = 6.0368 \pm 0.0186\ \frac{N}{m}$, $k_2 = 31.188 \pm 0.448\ \frac{N}{m}$, $k_4 = 53.64 \pm 0.66\ \frac{N}{m}$, and last $k_3 = 163.29 \pm 0.61\ \frac{N}{m}$.

From the even mode trials, where the pendulums are effectively uncoupled, the angular frequency of the pendulums, ω_0 was measured and found using a Fourier transform on the date collected. Using a Gaussian regression on the Fourier Transform data, the peak frequency was able to be determined and all the values averaged over the 16 trials and was found to be $4.171126 \pm 0.023111\ \frac{rad}{sec}$. The calculated error ranged from 0-1.5% of the theoretically calculated value of $4.19\ \frac{rad}{sec}$. From the odd mode trials, where the pendulums coupled, the angular frequency of the pendulums, ω was measured and found using a Fourier transform on the date collected. Using a gaussian regression on the Fourier Transform data, the peak frequency was able to be determined, then $\omega^2 vs k$ was plotted in Fig.6, keeping the mounting location ℓ constant for the 4 data sets plotted, as well as $\omega^2 vs \ell^2$ in Fig.7 plotted keeping the spring constant k constant for the 4 data sets plotted. The lines of best fit were found, and the slopes were compared to the theoretical values. From the graphs of $\omega^2 vs k$ in Fig.6 and $\omega^2 vs \ell^2$ in Fig.7, comparing to theoretical values there was a significant amount of error, ranging from about 8.0% to as much as 45%. During the experiment, there was some unstable oscillations transverse to the normal motion of the pendulum, due to the stiffness of the spring and unstable mounting (the spring just hooked onto the mounting locations rather than being fixed strongly to the bracket) allowing for bumping and slipping actions during the oscillations. Doing more trials with more spring may have been able to help with the regression to give a more accurate set of points for the line of best fit.

By using Fourier Analysis on the trials to find the beat frequency of the system, we were able to calculate, for two separate trials using a weak spring and a medium strength spring. Using by calculating theoretical values using $\Delta\omega = \omega_1 - \omega_2$ as the second term in Eq.8, the % Error was quite small (0.3%) for the trial with the weaker spring, but was larger with the medium strength spring (12.8%).

VII. REFERENCES

Jeff Gardiner. Coupled Oscillators. Waterloo, Ontario: University of Waterloo; c2014. 4 p.