Fourier Waveform Analysis

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Section 1
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I. ABSTRACT

The Fourier Series is an extremely useful tool in the analysis of periodic signals. This report presents a comparison between theoretically and experimentally determined values for the Fourier coefficients of eight common waveforms. The amplitudes are found to differ between 0% and 18.8%. It is understood that some of the discrepancies found between the two sets of values come from fluctuations and the approximations made in the signal generator to create the waveforms (especially for coefficients which were relatively small at higher harmonics compared to the fundamental harmonics), and through the rectification process used on some of the waveforms.

II. INTRODUCTION

The core of Fourier signal analysis lies in the fact that all functions exhibiting periodic behaviour over time can be decomposed into sums of sine and cosine functions. This property greatly simplifies the analysis of complicated waveforms, and allows for straightforward identification of properties such as harmonic frequencies and normal modes present in signals. This is discussed in detail in paper [5] on the use of Fourier Transforms in analyzing periodic functions. This paper, however, will focus on the closely related method of Fourier Series, with applications to a variety of common waveforms. The so-called Fourier coefficients of each signal will be calculated, and compared to their experimental counter-parts.

III. THEORETICAL BACKGROUND

Any continuous, periodic function $f(t)$, with period T has the property that:

$$\int_0^T f(t)dt = \int_k^{k+T} f(t)dt \tag{1}$$

Where k is any real valued constant.
We then work under the assumption that it can be expressed in the form:

$$f(t) = \sum_{n=0}^{\infty} a_n \cos(nt) + b_n \sin(nt) \tag{2}$$

This is motivated in part by the theory of Fourier Transforms. Now, we compute the a_n coefficients by multiplying the above expression by $\cos(mt)$, where m is an index like n, and integrating over t.

$$\int_0^{2\pi} f(t)\cos(mt)dt = \sum_{n=0}^{\infty}[a_n \int_0^{2\pi} \cos(nt)\cos(mt)dt + b_n \int_0^{2\pi} sin(nt)\cos(mt)dt] \tag{3}$$

Since sin and cos are odd and even respectively, the second term will disappear. Furthermore, by the orthogonality of cosines, the first term will only be non-zero under the condition that $n = m$. This allows us to write:

$$a_n \int_0^{2\pi} \cos(mt)\cos(mt)dt = \int_0^{2\pi} f(t)\cos(mt)dt \tag{4}$$

So,

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)dt \tag{5}$$

Since $\cos(nt) = 1$ at $n = 0$, we can write:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(t)dt \tag{6}$$

Similarly, for the b_n coefficients, we multiply by sin instead of cosine. Once again the odd-even integration rule is used along with the orthogonality of sines to obtain the expression:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(nt)dt \tag{7}$$

In this case, however, $b_0 = 0$ since $\sin(nt) = 0$ at $n = 0$.

With this last piece of information, it is possible to compute the Fourier Series of the periodic signals that will be analyzed in this paper.

IV. EXPERIMENTAL DESIGN AND PROCEDURE

A computer fitted with an oscilloscope card, and running a LabView program was used to acquire data on the eight waveforms examined in this experiment, and run peak analysis on the Fourier spectrum. These waveforms were created by an external signal generator, which was hooked up to an oscilloscope for visual confirmation of the signal. By hooking up the leads from the generator to the computer, it was possible to create eight waveforms through differentiation and rectification of the signal. These were:

1. Square Wave
2. Sawtooth (right angle triangle) Wave
3. Triangle (pyramid) Wave
4. Dirac Delta Signal
5. Alternating Polarity Dirac Delta Signal
6. Half-Wave Rectified Sine Wave
7. Half-Wave Rectified Sawtooth Wave
8. Full-Wave Rectified Sine Wave

V. ANALYSIS

For calculation purposes, all waves were assumed to have a maximum amplitude of 1, except that of the Up-Up and Up-Down Dirac Delta waves which by definition, have spikes of infinite amplitude.

A. Square Wave

We will start with deriving the values for the coefficients of the Fourier Series for a square wave, a_0, a_n, and b_n. Using (6), (5), and (7):

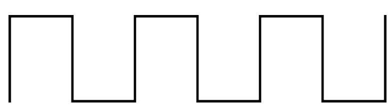

FIG. 1: Representation of a Square wave, Ref. [1]

$$
\begin{gathered}
a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)dt \\
a_0 = \frac{1}{\pi}[\int_0^{\pi} 1\cdot dt + \int_{\pi}^{2\pi} -1\cdot dt] \\
a_0 = \frac{1}{\pi}[t|_0^{\pi} - t|_{\pi}^{2\pi}] \\
a_0 = \frac{1}{\pi}[\pi - 0 - 2\pi + \pi] \\
a_0 = 0
\end{gathered}
$$

$$
\begin{gathered}
a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)dt \\
a_n = \frac{1}{\pi}[\int_0^{\pi} 1\cdot\cos(nt)dt + \int_{\pi}^{2\pi} -1\cdot\cos(nt)dt] \\
a_n = \frac{1}{\pi}[\frac{1}{n}\sin(nt)|_0^{\pi} - \frac{1}{n}\sin(nt)|_{\pi}^{2\pi}] \\
a_n = \frac{1}{\pi}\frac{1}{n}[\cancelto{0}{\sin(n\pi)} - \cancelto{0}{\sin 0} - \cancelto{0}{\sin(2n\pi)} + \cancelto{0}{\sin(n\pi)}] \\
a_n = 0
\end{gathered}
$$

$$
\begin{gathered}
b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(nt)dt \\
b_n = \frac{1}{\pi}[\int_0^{\pi} 1\cdot\sin(nt)dt + \int_{\pi}^{2\pi} -1\cdot\sin(nt)dt] \\
b_n = \frac{1}{\pi}[-\frac{1}{n}\cos(nt)|_0^{\pi} + \frac{1}{n}\cos(nt)|_{\pi}^{2\pi}] \\
b_n = \frac{1}{\pi}\frac{1}{n}[-\cos(n\pi) + \cancelto{1}{\cos(0)} + \cancelto{1}{\cos(2n\pi)} - \cos(n\pi)] \\
b_n = \frac{1}{\pi}\frac{2}{n}[1 + (-1)^n] \\
b_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{4}{n\pi}, & \text{if } n \text{ is odd} \end{cases}
\end{gathered}
$$

We can then determine that the Fourier Series for a Square wave to be:

$$
f(t) = \sum_{n\,odd}^{\infty} \frac{4}{n\pi}\sin(nt) \tag{8}
$$

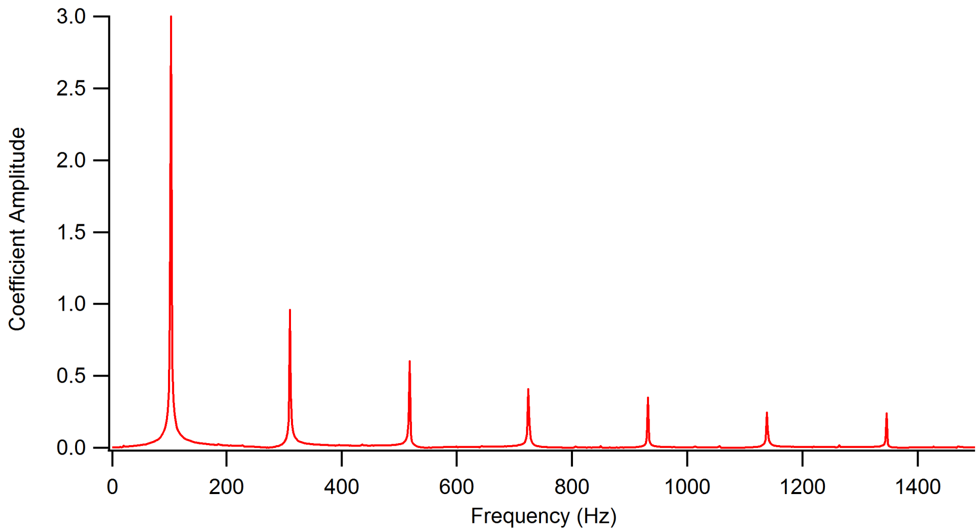

FIG. 2: Recorded line spectra of the generated 100 Hz Square Wave

Measured Line Spectra		Fourier Coefficients				Normalized	Percent
Frequency (Hz)	Amplitude	n	a_n	b_n	$c_n = \sqrt{a_n^2 + b_n^2}$	Amplitude	Difference
103.8	1.4423	1	0	1.2732	1.2732	1.2732	–
310.3	0.4952	3	0	0.4244	0.4244	0.4371	3.0
517.8	0.2871	5	0	0.2546	0.2546	0.2535	0.5
724.3	0.2158	7	0	0.1819	0.1819	0.1905	4.7
932.1	0.1555	9	0	0.1415	0.1415	0.1373	3.0
1138.5	0.1380	11	0	0.1157	0.1157	0.1218	5.2
1346.0	0.1045	13	0	0.0979	0.0979	0.0923	5.8

TABLE I: Square wave measured data and comparison to theoretical values.

The measured coefficients for the Square wave taken from the line spectrum graph in Fig.2 once normalized with the first fundamental term agree with the calculated theoretical Fourier series coefficients, only fluctuating from 0.5% to 5.8% difference between the 1^{st} and 13^{th} harmonics. The frequency locations of the coefficients also agree with the theory where they are only present on odd harmonics as defined in the Fourier series for a Square wave in (8). The frequencies of the harmonics are, agree with only slight variation with the theoretical value of 100Hz output by the waveform generator. Some of the discrepancies may be from the slight flaws or variation in timings made by the waveform generator or background noise in the waveform output which may skew results unpredictably.

Sample Calculations for b_n using Table I row 1, for $n = 1$

$$b_n = \frac{4}{n\pi}$$
$$b_1 = \frac{4}{\pi}$$
$$b_1 = 1.2732$$

Sample Calculations for $NormalizedAmplitude$ using Table I row 2

$$A_{n_{normalized}} = A \times \frac{c_1}{A_1}$$
$$A_{n_{normalized}} = 0.4952 \times \frac{1.2732}{1.4423}$$
$$A_{n_{normalized}} = 0.4371$$

Sample Calculations for $PercentDifference$ using Table I row 2

$$\%_{Diff} = \frac{|c_n - A_{n_{normalized}}|}{c_n} \times 100\%$$
$$\%_{Diff} = \frac{|0.4244 - 0.4371|}{0.4244} \times 100\%$$
$$\%_{Diff} = 2.99\%$$
$$\%_{Diff} \approx 3.0\%$$

B. Sawtooth Wave

FIG. 3: Representation of a Sawtooth wave, Ref. [1]

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{n\pi}$$

The Fourier Series for a Sawtooth wave is then:

$$f(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin(nt) \tag{9}$$

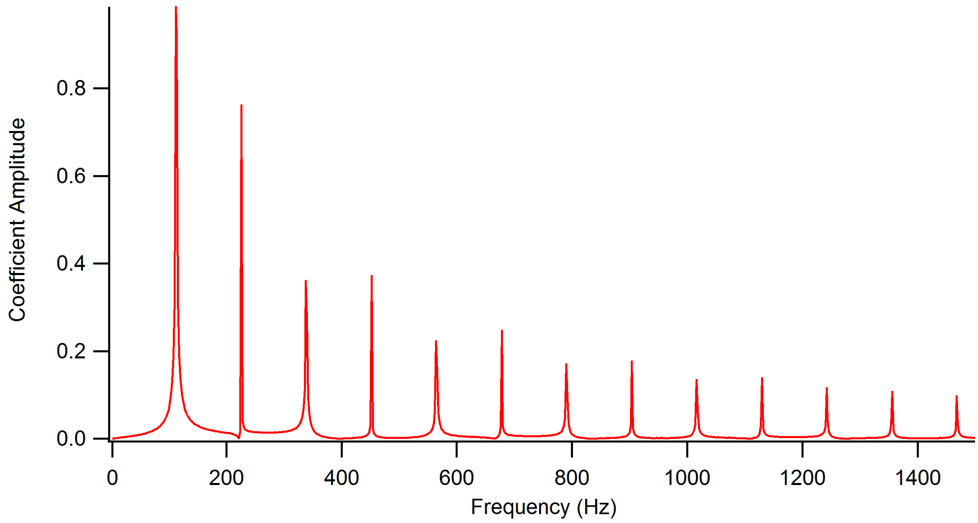

FIG. 4: Recorded line spectra of the generated 100 Hz Sawtooth Wave

Measured Line Spectra		Fourier Coefficients				Normalized	Percent
Frequency (Hz)	Amplitude	n	a_n	b_n	$c_n = \sqrt{a_n^2 + b_n^2}$	Amplitude	Difference
113.0	0.9239	1	0	0.6366	0.6366	0.6366	–
226.0	0.4641	2	0	0.3183	0.3183	0.3198	0.5
338.7	0.3058	3	0	0.2122	0.2122	0.2107	0.7
452.0	0.2276	4	0	0.1592	0.1592	0.1568	1.5
564.7	0.1791	5	0	0.1273	0.1273	0.1234	3.1
678.1	0.1516	6	0	0.1061	0.1061	0.1045	1.5
790.4	0.1263	7	0	0.0909	0.0909	0.0870	4.3
903.9	0.1096	8	0	0.0796	0.0796	0.0755	5.1
1016.5	0.0946	9	0	0.0707	0.0707	0.0652	7.9
1130.0	0.0864	10	0	0.0637	0.0637	0.0595	6.5
1242.2	0.0769	11	0	0.0579	0.0579	0.0530	8.5
1355.8	0.0675	12	0	0.0531	0.0531	0.0465	12.3
1468.2	0.0615	13	0	0.0490	0.0490	0.0424	13.5

TABLE II: Sawtooth wave measured data and comparison to theoretical values.

The measured coefficients for the Sawtooth wave taken from the line spectrum graph in Fig.4 once normalized with the first fundamental term agree with the calculated theoretical Fourier series coefficients, only fluctuating from 0.5% to 13.5% difference between the 1^{st} and 13^{th} harmonics with increasing error at higher harmonics. This increasing error may be due to the decreasing amplitude of the coefficients which could then be more prone to any fluctuations as a small change in the measured coefficient amplitude in the Sawtooth wave line spectrum (Fig.4) and these changes would be relatively large for smaller amplitudes. The frequency locations of the coefficients also agree with the theory if the waveform generator was outputting a the Sawtooth wave with a period of approximately 113 Hz as the data suggests.

Sample Calculations for b_n using Table II row 1, for $n = 1$

$$b_n = \frac{2}{n\pi}$$
$$b_1 = \frac{2}{\pi}$$
$$b_1 = 0.6366$$

Sample Calculations for $NormalizedAmplitude$ using Table II row 2

$$A_{n_{normalized}} = A \times \frac{c_1}{A_1}$$
$$A_{n_{normalized}} = 0.464 \times \frac{0.6366}{0.924}$$
$$A_{n_{normalized}} = 0.3198$$

Sample Calculations for $PercentDifference$ using Table II row 2

$$\%_{Diff} = \frac{|c_n - A_{n_{normalized}}|}{c_n} \times 100\%$$
$$\%_{Diff} = \frac{|0.3183 - 0.3198|}{0.3183} \times 100\%$$
$$\%_{Diff} = 0.47\%$$
$$\%_{Diff} \approx 0.5\%$$

C. Triangle Wave

FIG. 5: Representation of a Triangle wave, Ref. [1]

$$a_0 = 0$$

$$a_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{8}{(n\pi)^2}, & \text{if } n \text{ is odd} \end{cases}$$

$$b_n = 0$$

The Fourier Series for a Triangle wave is then:

$$f(t) = \sum_{n\,odd}^{\infty} \frac{8}{(n\pi)^2} \cos(nt) \tag{10}$$

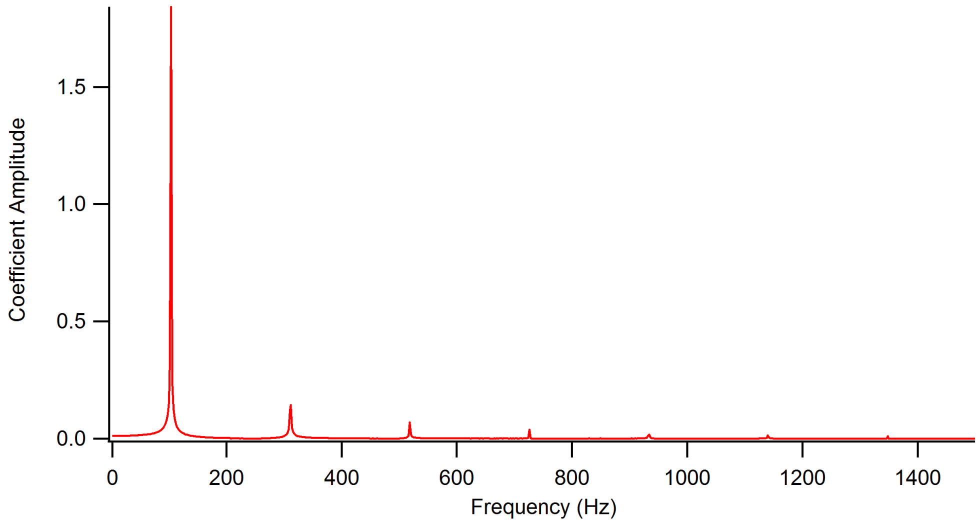

FIG. 6: Recorded line spectra of the generated 100 Hz Triangle Wave

Measured Line Spectra		Fourier Coefficients				Normalized	Percent
Frequency (Hz)	Amplitude	n	a_n	b_n	$c_n = \sqrt{a_n^2 + b_n^2}$	Amplitude	Difference
103.9	0.7568	1	0.8106	0	0.8106	0.8106	–
311.2	0.0915	3	0.0901	0	0.0901	0.0980	8.8
518.2	0.0318	5	0.0324	0	0.0324	0.0341	5.2
722.6	0.0147	7	0.0165	0	0.0165	0.0157	4.8
933.4	0.0100	9	0.0100	0	0.0100	0.0107	6.7
1140.2	0.0068	11	0.0067	0	0.0067	0.0073	8.8
1348.0	0.0036	13	0.0048	0	0.0048	0.0039	18.8

TABLE III: Triangle wave measured data and comparison to theoretical values.

The measured coefficients for the Triangle wave taken from the line spectrum graph in Fig.6 once normalized with the first fundamental term agree with the calculated theoretical Fourier series coefficients, but are affected by the small coefficients due to the $\frac{1}{n^2}$ relationship differing by 4.8% to 18.8%. The error is the same as that mentioned previously, with the granularity of the measurements, small fluctuations affect small amplitude co effects to a larger degree. The frequency locations of the coefficients also agree with the waveform generator theoretical output frequency of close to 100Hz.

Sample Calculations for a_n using Table III row 1, for $n = 1$

$$a_n = \frac{8}{(n\pi)^2}$$
$$a_1 = \frac{8}{(\pi)^2}$$
$$a_1 = 0.8106$$

Sample Calculations for $NormalizedAmplitude$ using Table III row 2

$$A_{n_{normalized}} = A \times \frac{c_1}{A_1}$$
$$A_{n_{normalized}} = 0.0915 \times \frac{0.8106}{0.7568}$$
$$A_{n_{normalized}} = 0.0980$$

Sample Calculations for $PercentDifference$ using Table III row 2

$$\%_{Diff} = \frac{|c_n - A_{n_{normalized}}|}{c_n} \times 100\%$$
$$\%_{Diff} = \frac{|0.0901 - 0.0980|}{0.0901} \times 100\%$$
$$\%_{Diff} = 8.76\%$$
$$\%_{Diff} \approx 8.8\%$$

D. Up-Up Dirac Delta Wave, 2π-periodic

FIG. 7: Representation of a Up-Up Dirac Delta wave, also known as Dirac's Comb, Ref. [1]

$$a_0 = \frac{1}{\pi}$$

$$a_n = \frac{2}{\pi}$$

$$b_n = 0$$

The Fourier Series for a Up-Up Dirac Delta wave is then:

$$f(t) = \frac{1}{\pi} + \sum_{n=1}^{\infty} \frac{2}{\pi}\cos(nt) \tag{11}$$

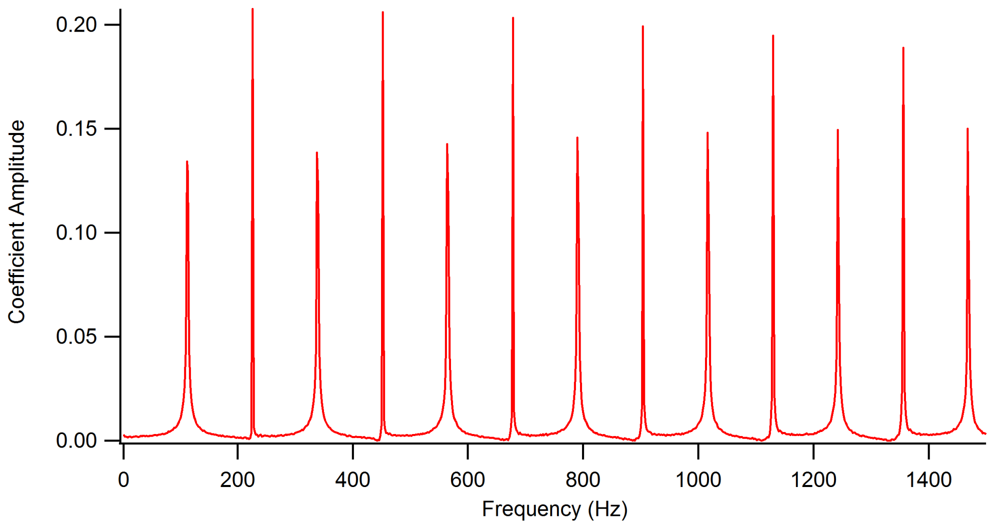

FIG. 8: Recorded line spectra of the generated 100 Hz Up-Up Dirac Delta Wave

Measured Line Spectra		Fourier Coefficients				Normalized	Percent
Frequency (Hz)	Amplitude	n	a_n	b_n	$c_n = \sqrt{a_n^2 + b_n^2}$	Amplitude	Difference
113.0	0.1260	1	0.6366	0	0.6366	0.6366	–
226.1	0.1258	2	0.6366	0	0.6366	0.6356	0.2
338.9	0.1252	3	0.6366	0	0.6366	0.6328	0.6
451.9	0.1248	4	0.6366	0	0.6366	0.6306	0.9
564.8	0.1243	5	0.6366	0	0.6366	0.6278	1.4
678.0	0.1319	6	0.6366	0	0.6366	0.6662	4.6
790.8	0.1227	7	0.6366	0	0.6366	0.6199	2.6
904.0	0.1287	8	0.6366	0	0.6366	0.6504	2.2
1016.7	0.1208	9	0.6366	0	0.6366	0.6104	4.1
1129.5	0.1296	10	0.6366	0	0.6366	0.6547	2.8
1242.6	0.1285	11	0.6366	0	0.6366	0.6490	1.9
1356.3	0.1267	12	0.6366	0	0.6366	0.6403	0.6
1468.5	0.1257	13	0.6366	0	0.6366	0.6353	0.2

TABLE IV: Up-Up Dirac Delta wave measured data and comparison to theoretical values.

The measured coefficients for the Up-Up Dirac Delta taken from the line spectrum graph in Fig.8 once normalized with the first fundamental term agree with the calculated theoretical Fourier series coefficients, quite well, only differing between 0.2% to 4.6% quite consistently. Any discrepancy could be due to the fact that the generated wave is not a perfect Dirac Delta function at its pulses, but they have some defined hight and width which may change the coefficients to agree closer to that of a pulse train Fourier series representation.

Sample Calculations for a_n using Table IV row 1, for $n = 1$

$$a_n = \tfrac{2}{\pi}$$
$$a_1 = \tfrac{2}{\pi}$$
$$a_1 = 0.6366$$

Sample Calculations for $NormalizedAmplitude$ **using Table IV row 2**

$$A_{n_{normalized}} = A \times \frac{c_1}{A_1}$$
$$A_{n_{normalized}} = 0.1258 \times \frac{0.6366}{0.1260}$$
$$A_{n_{normalized}} = 0.6356$$

Sample Calculations for $PercentDifference$ **using Table IV row 2**

$$\%_{Diff} = \frac{|c_n - A_{n_{normalized}}|}{c_n} \times 100\%$$
$$\%_{Diff} = \frac{|0.6366 - 0.6356|}{0.6366} \times 100\%$$
$$\%_{Diff} = 0.16\%$$
$$\%_{Diff} \approx 0.2\%$$

E. Up-Down Dirac Delta Wave, 2π-periodic

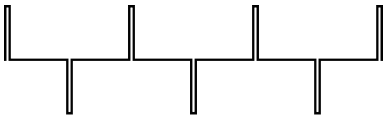

FIG. 9: Representation of a Up-Down Dirac Delta wave, Ref. [1]

$$a_0 = 0$$

$$a_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{4}{\pi}, & \text{if } n \text{ is odd} \end{cases}$$

$$b_n = 0$$

The Fourier Series for a Up-Down Dirac Delta wave is then:

$$f(t) = \sum_{n\, odd}^{\infty} \frac{4}{\pi} \cos(nt) \tag{12}$$

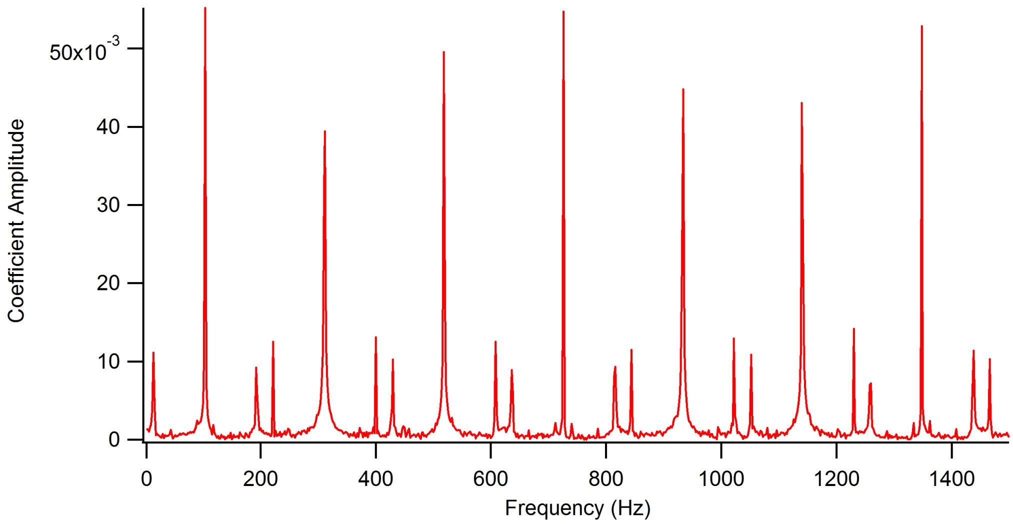

FIG. 10: Recorded line spectra of the generated 100 Hz Up-Down Dirac Delta Wave

Measured Line Spectra		Fourier Coefficients				Normalized	Percent
Frequency (Hz)	Amplitude	n	a_n	b_n	$c_n = \sqrt{a_n^2 + b_n^2}$	Amplitude	Difference
103.9	0.0326	1	1.2732	0	1.2732	1.2732	–
311.2	0.0333	3	1.2732	0	1.2732	1.3005	2.1
518.3	0.0335	5	1.2732	0	1.2732	1.3095	2.9
726.0	0.0329	7	1.2732	0	1.2732	1.2875	1.1
933.5	0.0326	9	1.2732	0	1.2732	1.2737	0.0
1140.5	0.0325	11	1.2732	0	1.2732	1.2710	0.2
1348.0	0.0322	13	1.2732	0	1.2732	1.2570	1.3

TABLE V: Up-Down Dirac Delta wave measured data and comparison to theoretical values.

The measured coefficients for the Up-Down Dirac Delta wave taken from the line spectrum graph in Fig.10 once normalized with the first fundamental term agree with the calculated theoretical Fourier series coefficients also agrees quite well, only differing between 0.0% to 2.9% quite consistently. Any discrepancy could also, as for the Up-Up Dirac Delta wave, could be due to the fact that the generated wave is not a perfect Dirac Delta function at its pulses.

Sample Calculations for a_n using Table V row 1, for $n = 1$

$$a_n = \frac{4}{\pi}$$
$$a_1 = \frac{4}{\pi}$$
$$a_1 = 1.2732$$

Sample Calculations for $NormalizedAmplitude$ using Table V row 2

$$A_{n_{normalized}} = A \times \frac{c_1}{A_1}$$
$$A_{n_{normalized}} = 0.0333 \times \frac{1.2732}{0.0326}$$
$$A_{n_{normalized}} = 1.3005$$

Sample Calculations for $PercentDifference$ using Table V row 2

$$\%_{Diff} = \frac{|c_n - A_{n_{normalized}}|}{c_n} \times 100\%$$
$$\%_{Diff} = \frac{|1.2732 - 1.3005|}{1.2732} \times 100\%$$
$$\%_{Diff} = 2.14\%$$
$$\%_{Diff} \approx 2.1\%$$

F. Half-Wave Rectified Sine Wave

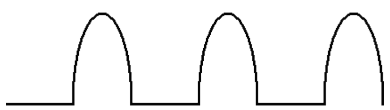

FIG. 11: Representation of a Half-Wave Rectified Sine wave, Ref. [1]

$$a_0 = \frac{1}{\pi}$$

$$a_n = \begin{cases} \frac{2}{\pi(1-n^2)}, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

$$b_1 = \frac{1}{2}$$

$$b_n = 0$$

The Fourier Series for a Half-Wave Rectified Sine wave is then:

$$f(t) = \frac{1}{\pi} + \frac{1}{2} sin(t) + \sum_{n\,even}^{\infty} \frac{2}{\pi(1-n^2)} cos(nt) \tag{13}$$

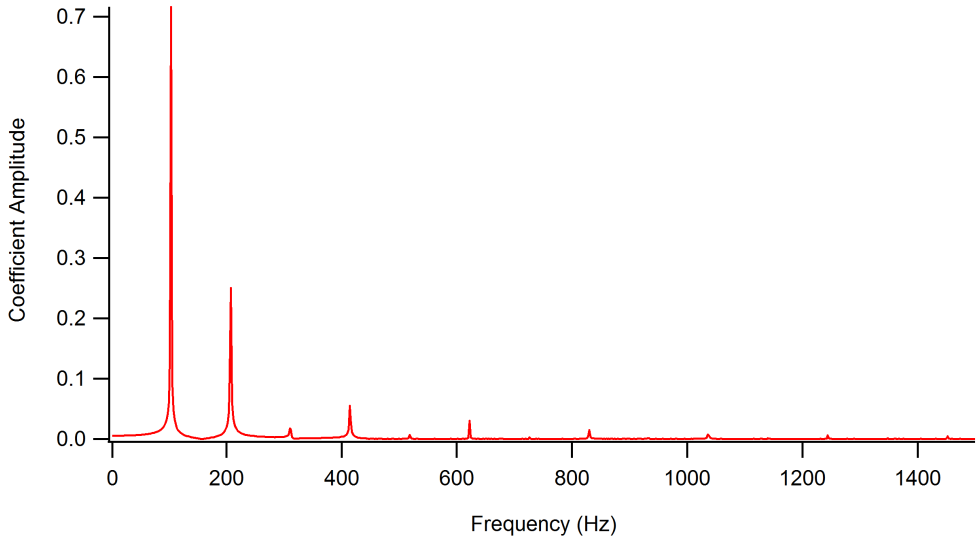

FIG. 12: Recorded line spectra of the generated 100 Hz Half-Wave Rectified Sine Wave

Measured Line Spectra		Fourier Coefficients				Normalized	Percent
Frequency (Hz)	Amplitude	n	a_n	b_n	$c_n = \sqrt{a_n^2 + b_n^2}$	Amplitude	Difference
103.9	0.6210	1	0	0.5	0.5	0.5	–
207.7	0.2420	2	-0.2122	0	0.2122	0.1948	8.2
414.5	0.0540	4	-0.0424	0	0.0424	0.0435	2.4
622.1	0.0227	6	-0.0182	0	0.0182	0.0183	0.5
829.5	0.0132	8	-0.0101	0	0.0101	0.0106	5.2
1037.0	0.0071	10	-0.0064	0	0.0064	0.0057	11.1
1244.340	0.0054	12	-0.0045	0	0.0045	0.0043	2.3
1452.260	0.0037	14	-0.0033	0	0.0033	0.0030	8.7

TABLE VI: Half-Wave Rectified Sine wave measured data and comparison to theoretical values.

The measured coefficients for the Half-Wave Rectified Sine wave taken from the line spectrum graph in Fig.12 once normalized with the first fundamental term agree with the calculated theoretical Fourier series coefficients, only fluctuating from 0.5% to 11.1%. The frequency locations of the coefficients also agree with the theory where they are only present on even harmonics and the the first harmonic as defined in the Fourier series for a Half-Wave Rectified Sine wave in (13). There are some noticeable, yet extremely tiny peaks on odd harmonic locations in the Line Spectrum Fig. 12 which may be due to small inaccuracies in the waveform near the boundary of the function.

Sample Calculations for a_n using Table VI row 1, for $n = 2$

$$a_n = \frac{2}{\pi(1-n^2)}$$
$$a_2 = \frac{2}{\pi(1-2^2)}$$
$$a_2 = -0.2122$$

Sample Calculations for $NormalizedAmplitude$ using Table VI row 2

$$A_{n_{normalized}} = A \times \frac{c_1}{A_1}$$
$$A_{n_{normalized}} = 0.2420 \times \frac{0.5}{0.6210}$$
$$A_{n_{normalized}} = 0.1948$$

Sample Calculations for $PercentDifference$ using Table VI row 2

$$\%_{Diff} = \frac{|c_n - A_{n_{normalized}}|}{c_n} \times 100\%$$
$$\%_{Diff} = \frac{|0.2122 - 0.1948|}{0.2122} \times 100\%$$
$$\%_{Diff} = 8.1998\%$$
$$\%_{Diff} \approx 8.2\%$$

G. Half-Wave Rectified Sawtooth Wave

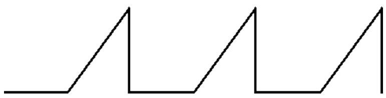

FIG. 13: Representation of a Half-Wave Rectified Sawtooth wave, Ref. [1]

$$a_0 = \frac{1}{4}$$

$$a_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{-2}{(n\pi)^2}, & \text{if } n \text{ is odd} \end{cases}$$

$$b_n = \frac{(-1)^{n+1}}{n\pi}$$

The Fourier Series for a Half-Wave Rectified Sawtooth wave is then:

$$f(t) = \frac{1}{4} + \sum_{n\,odd}^{\infty} \frac{-2}{(n\pi)^2} cos(nt) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} sin(nt) \tag{14}$$

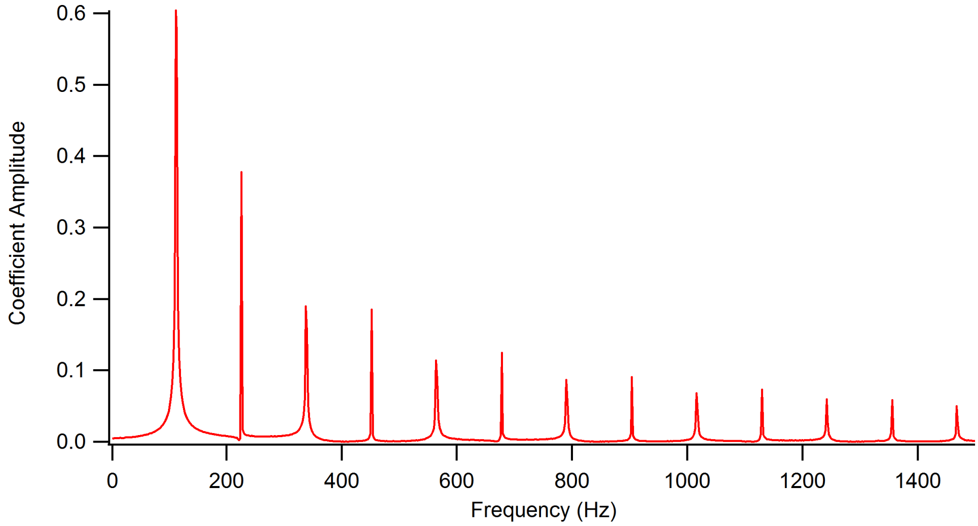

FIG. 14: Recorded line spectra of the generated 100 Hz Half-Wave Rectified Sawtooth Wave

Measured Line Spectra		Fourier Coefficients				Normalized	Percent
Frequency (Hz)	Amplitude	n	a_n	b_n	$c_n = \sqrt{a_n^2 + b_n^2}$	Amplitude	Difference
113.0	0.5733	1	-0.2026	0.3183	0.3773	0.3583	–
226.0	0.2411	2	0	-0.1592	0.1592	0.1507	5.3
338.7	0.1650	3	-0.0225	0.1061	0.1085	0.1031	4.9
452.0	0.1173	4	0	-0.0796	0.0796	0.0733	7.9
564.8	0.0953	5	-0.0081	0.0637	0.0642	0.0596	7.1
678.0	0.0779	6	0	-0.0531	0.0531	0.0487	8.3
790.5	0.0673	7	-0.0041	0.0455	0.0457	0.0420	7.9
904.0	0.0650	8	0	-0.0398	0.0398	0.0406	2.1
1016.6	0.0512	9	-0.0025	0.0354	0.0355	0.0320	9.8
1130.0	0.0466	10	0	-0.0318	0.0318	0.0291	8.4
1242.3	0.0420	11	-0.0017	0.0289	0.0290	0.0263	9.4
1355.9	0.0400	12	0	-0.0265	0.0265	0.0250	5.8
1468.4	0.0382	13	-0.0012	0.0245	0.0245	0.0239	2.6

TABLE VII: Half-Wave Rectified Sawtooth wave measured data and comparison to theoretical values.

The measured coefficients for the Half-Wave Rectified Sawtooth wave taken from the line spectrum graph in Fig.14 once normalized with the first fundamental term agree with the calculated theoretical Fourier series coefficients, only fluctuating from 2.1% to 9.8%. The fundamental frequency agrees with the our data from part B of the lab, as the non-rectified sawtooth wave also had a fundamental frequency of 113Hz which is close to the specified 100Hz from the waveform generator.

Sample Calculations for a_n and b_n using Table VII row 1, for $n = 1$

$$a_n = \frac{-2}{(n\pi)^2}$$
$$a_1 = \frac{-2}{(\pi)^2}$$
$$a_1 = -0.2026$$

$$b_n = \frac{(-1)^{n+1}}{n\pi}$$
$$b_1 = \frac{(-1)^1}{\pi}$$
$$b_1 = 0.3183$$

Sample Calculations for $NormalizedAmplitude$ using Table VII row 2

$$A_{n_{normalized}} = A \times \frac{c_1}{A_1}$$
$$A_{n_{normalized}} = 0.2411 \times \frac{0.3583}{0.5733}$$
$$A_{n_{normalized}} = 0.1507$$

Sample Calculations for $PercentDifference$ using Table VII row 2

$$\%_{Diff} = \frac{|c_n - A_{n_{normalized}}|}{c_n} \times 100\%$$
$$\%_{Diff} = \frac{|0.1592 - 0.1507|}{0.1592} \times 100\%$$
$$\%_{Diff} = 5.34\%$$
$$\%_{Diff} \approx 5.3\%$$

H. Full-Wave Rectified Sine Wave

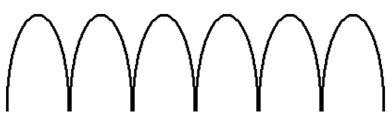

FIG. 15: Representation of a Full-Wave Rectified Sine wave, Ref. [1]

$$a_0 = \frac{2}{\pi}$$

$$a_n = \begin{cases} \frac{4}{\pi(1-n^2)}, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

$$b_n = 0$$

The Fourier Series for a Full-Wave Rectified Sine wave is then:

$$f(t) = \frac{2}{\pi} + \sum_{n\,even}^{\infty} \frac{4}{\pi(1-n^2)} cos(nt) \tag{15}$$

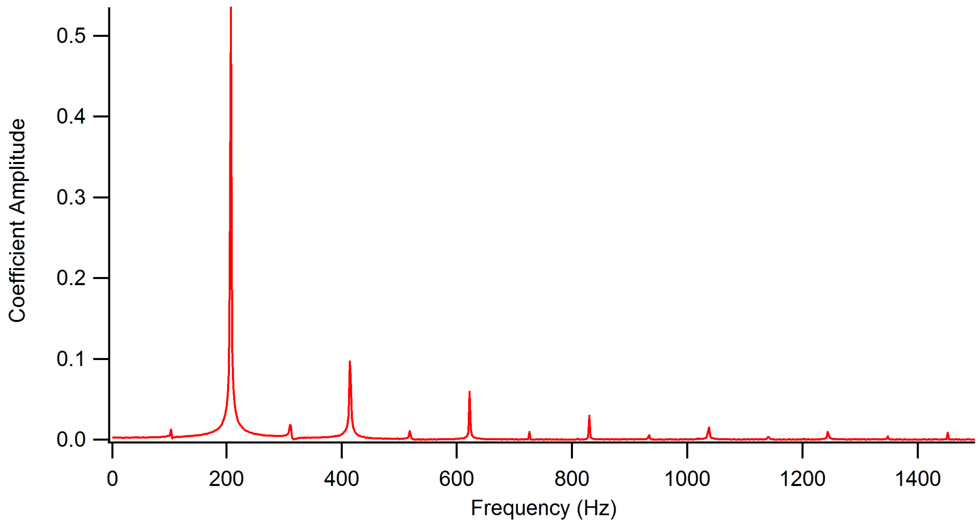

FIG. 16: Recorded line spectra of the generated 100 Hz Full-Wave Rectified Sine Wave

Measured Line Spectra		Fourier Coefficients				Normalized	Percent
Frequency (Hz)	Amplitude	n	a_n	b_n	$c_n = \sqrt{a_n^2 + b_n^2}$	Amplitude	Difference
207.6	0.4700	2	-0.4244	0	0.4244	0.4244	–
414.8	0.0900	4	-0.0849	0	0.0849	0.0813	4.2
622.2	0.0450	6	-0.0364	0	0.0364	0.0406	11.7
830.0	0.0230	8	-0.0202	0	0.0202	0.0208	2.8
1037.4	0.0138	10	-0.0129	0	0.0129	0.0124	3.5
1244.2	0.0090	12	-0.0089	0	0.0089	0.0081	8.7
1452.1	0.0072	14	-0.0065	0	0.0065	0.0065	0.4

TABLE VIII: Full-Wave Rectified Sine wave measured data and comparison to theoretical values.

The measured coefficients for the Full-Wave Rectified Sine wave taken from the line spectrum graph in Fig.16 once normalized with the first fundamental term agree with the calculated theoretical Fourier series coefficients, only fluctuating from 0.4% to 11.7%. The frequency locations of the coefficients also agree with the theory where they are only present on even harmonics as defined in the Fourier series for a Full-Wave Rectified Sine wave in (15). There are some noticeable, yet extremely tiny peaks on odd harmonic locations in the Line Spectrum Fig. 16 as there were for the Half-Wave Rectified Sine wave which may be due to small inaccuracies in the waveform near the boundaries of the function as before.

Sample Calculations for a_n using Table VIII row 1, for $n = 2$

$$a_n = \frac{4}{\pi(1-n^2)}$$
$$a_1 = \frac{4}{\pi(1-2^2)}$$
$$a_1 = -0.4244$$

Sample Calculations for $NormalizedAmplitude$ using Table VIII row 2

$$A_{n_{normalized}} = A \times \frac{c_1}{A_1}$$
$$A_{n_{normalized}} = 0.0900 \times \frac{0.4244}{0.4700}$$
$$A_{n_{normalized}} = 0.0813$$

Sample Calculations for $PercentDifference$ using Table VIII row 2

$$\%_{Diff} = \frac{|c_n - A_{n_{normalized}}|}{c_n} \times 100\%$$
$$\%_{Diff} = \frac{|0.0849 - 0.0813|}{0.0849} \times 100\%$$
$$\%_{Diff} = 4.24\%$$
$$\%_{Diff} \approx 4.2\%$$

VI. CONCLUSION

The analysis of the waves A though H and the deconstruction of the fundamental frequencies into the Fourier Series coefficients using the LabView software for each wave allowed us to compare the measured values with the theoretical Fourier coefficients for each respective wave. We were able to confirm the theory as the amplitude, once normalized, agreed with the theoretical values with a difference of 0% to 18.8%. For most coefficients in all the waves the difference was less than 10%, large errors were only incurred at harmonics that displayed a small Fourier coefficient, the precision of these harmonics were susceptible to larger relative errors due to small fluctuations which were present in the wave analysis. The fundamental frequency of the waves was also in line with the specified frequency of 100Hz on the waveform generator, with all the waves having a fundamental frequency of about 103Hz, and the Sawtooth wave having a fundamental frequency of 113Hz. Waveforms with Fourier coefficients that were zero at either odd or even harmonics was also confirmed through the data collected for waves such as the Square wave, Triangle wave, Up-Down Dirac Delta wave, Half-Wave Rectified Sine wave, and the Full-Wave Rectified Sine wave.

Some discrepancies with some small peaks appearing in harmonics which should not be present were notices in the Up-Down Dirac Delta wave, the Half-Wave Rectified Sine wave, and the Full-Wave Rectified Sine wave which can be due to some inaccuracies in the generated waveform. the pulses in the Dirac Delta waveforms are only approximations which could lead to some other harmonics. The rectification process of the Sine wave may also incur some unknown inaccuracies, the waveform generator would need to be studied in more depth to ascertain any factors contributing to these harmonics shown on the Line Spectrum graphs.

VII. REFERENCES

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