CHAPTER 17

Fourier Method of Waveform Analysis

17.1 INTRODUCTION

In the circuits examined previously, the response was obtained for excitations having constant, sinusoidal, or exponential form. In such cases a single expression described the forcing function for all time; for instance, $v = \text{constant}$ or $v = V \sin \omega t$, as shown in Fig. 17-1(*a*) and (*b*).

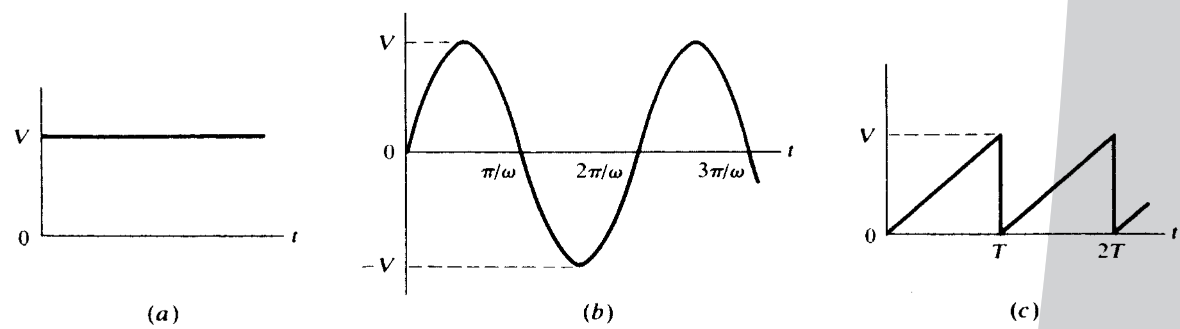

Fig. 17-1

Certain periodic waveforms, of which the sawtooth in Fig. 17-1(*c*) is an example, can be only locally defined by single functions. Thus, the sawtooth is expressed by $f(t) = (V/T)t$ in the interval $0 < t < T$ and by $f(t) = (V/T)(t - T)$ in the interval $T < t < 2T$. While such piecemeal expressions describe the waveform satisfactorily, they do not permit the determination of the circuit response. Now, if a periodic function can be expressed as the sum of a finite or infinite number of sinusoidal functions, the responses of linear networks to nonsinusoidal excitations can be determined by applying the superposition theorem. The Fourier method provides the means for solving this type of problem.

In this chapter we develop tools and conditions for such expansions. Periodic waveforms may be expressed in the form of Fourier series. Nonperiodic waveforms may be expressed by their Fourier transforms. However, a piece of a nonperiodic waveform specified over a finite time period may also be expressed by a Fourier series valid within that time period. Because of this, the Fourier series analysis is the main concern of this chapter.

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17.2 TRIGONOMETRIC FOURIER SERIES

Any periodic waveform—that is, one for which $f(t) = f(t+T)$—can be expressed by a Fourier series provided that

(1) If it is discontinuous, there are only a finite number of discontinuities in the period T;

(2) It has a finite average value over the period T;

(3) It has a finite number of positive and negative maxima in the period T.

When these *Dirichlet conditions* are satisfied, the Fourier series exists and can be written in trigonometric form:

$$f(t) = \tfrac{1}{2}a_0 + a_1 \cos \omega t + a_2 \cos 2t + a_3 \cos 3\omega t + \cdots \\ + b_1 \sin \omega t + b_2 \sin 2\omega t + b_3 \sin 3\omega t + \cdots \tag{1}$$

The Fourier coefficients, a's and b's, are determined for a given waveform by the evaluation integrals. We obtain the cosine coefficient evaluation integral by multiplying both sides of (*1*) by $\cos n\omega t$ and integrating over a full period. The period of the fundamental, $2\pi/\omega$, is the period of the series since each term in the series has a frequency which is an integral multiple of the fundamental frequency.

$$\int_0^{2\pi/\omega} f(t) \cos n\omega t \, dt = \int_0^{2\pi/\omega} \frac{1}{2} a_0 \cos n\omega t \, dt + \int_0^{2\pi/\omega} a_1 \cos \omega t \cos n\omega t \, dt + \cdots \\ + \int_0^{2\pi/\omega} a_n \cos^2 n\omega t \, dt + \cdots + \int_0^{2\pi/\omega} b_1 \sin \omega t \cos n\omega t \, dt \\ + \int_0^{2\pi/\omega} b_2 \sin 2\omega t \cos n\omega \, dt + \cdots \tag{2}$$

The definite integrals on the right side of (*2*) are all zero except that involving $\cos^2 n\omega t$, which has the value $(\pi/\omega)a_n$. Then

$$a_n = \frac{\omega}{\pi} \int_0^{2\pi/\omega} f(t) \cos n\omega t \, dt = \frac{2}{T} \int_0^T f(t) \cos \frac{2\pi n t}{T} \, dt \tag{3}$$

Multiplying (*1*) by $\sin n\omega t$ and integrating as above results in the sine coefficient evaluation integral.

$$b_n = \frac{\omega}{\pi} \int_0^{2\pi/\omega} f(t) \sin n\omega t \, dt = \frac{2}{T} \int_0^T f(t) \sin \frac{2\pi n t}{T} \, dt \tag{4}$$

An alternate form of the evaluation integrals with the variable $\psi = \omega t$ and the corresponding period 2π radians is

$$a_n = \frac{1}{\pi} \int_0^{2\pi} F(\psi) \cos n\psi \, d\psi \tag{5}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} F(\psi) \sin n\psi \, d\psi \tag{6}$$

where $F(\psi) = f(\psi/\omega)$. The integrations can be carried out from $-T/2$ to $T/2$, $-\pi$ to $+\pi$, or over any other full period that might simplify the calculation. The constant a_0 is obtained from (*3*) or (*5*) with $n = 0$; however, since $\frac{1}{2}a_0$ is the average value of the function, it can frequently be determined by inspection of the waveform. The series with coefficients obtained from the above evaluation integrals converges uniformly to the function at all points of continuity and converges to the mean value at points of discontinuity.

EXAMPLE 17.1 Find the Fourier series for the waveform shown in Fig. 17-2.

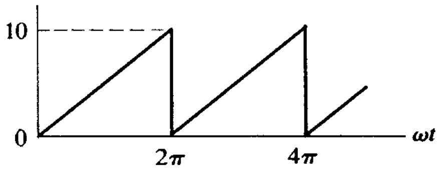


Fig. 17-2

The waveform is periodic, of period $2\pi/\omega$ in t or 2π in ωt. It is continuous for $0 < \omega t < 2\pi$ and given therein by $f(t) = (10/2\pi)\omega t$, with discontinuities at $\omega t = n2\pi$ where $n = 0, 1, 2, \ldots$. The Dirichlet conditions are satisfied. The average value of the function is 5, by inspection, and thus, $\frac{1}{2}a_0 = 5$. For $n > 0$, (*5*) gives

$$a_n = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{10}{2\pi}\right)\omega t \cos n\omega t \, d(\omega t) = \frac{10}{2\pi^2}\left[\frac{\omega t}{n}\sin n\omega t + \frac{1}{n^2}\cos n\omega t\right]_0^{2\pi}$$
$$= \frac{10}{2\pi^2 n^2}(\cos n2\pi - \cos 0) = 0$$

Thus, the series contains no cosine terms. Using (*6*), we obtain

$$b_n = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{10}{2\pi}\right)\omega t \sin n\omega t \, d(\omega t) = \frac{10}{2\pi^2}\left[-\frac{\omega t}{n}\cos n\omega t + \frac{1}{n^2}\sin n\omega t\right]_0^{2\pi} = -\frac{10}{\pi n}$$

Using these sine-term coefficients and the average term, the series is

$$f(t) = 5 - \frac{10}{\pi}\sin \omega t - \frac{10}{2\pi}\sin 2\omega t - \frac{10}{3\pi}\sin 3\omega t - \cdots = 5 - \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\omega t}{n}$$

The sine and cosine terms of like frequency can be combined as a single sine or cosine term with a phase angle. Two alternate forms of the trigonometric series result.

$$f(t) = \tfrac{1}{2}a_0 + \sum c_n \cos(n\omega t - \theta_n) \tag{7}$$

and

$$f(t) = \tfrac{1}{2}a_0 + \sum c_n \sin(n\omega t + \phi_n) \tag{8}$$

where $c_n = \sqrt{a_n^2 + b_n^2}$, $\theta_n = \tan^{-1}(b_n/a_n)$, and $\phi_n = \tan^{-1}(a_n/b_n)$. In (*7*) and (*8*), c_n is the harmonic amplitude, and the harmonic phase angles are θ_n or ϕ_n.

17.3 EXPONENTIAL FOURIER SERIES

A periodic waveform $f(t)$ satisfying the Dirichlet conditions can also be written as an exponential Fourier series, which is a variation of the trigonometric series. The exponential series is

$$f(t) = \sum_{n=-\infty}^{\infty} \mathbf{A}_n e^{jn\omega t} \tag{9}$$

To obtain the evaluation integral for the $\mathbf{A}_n$ coefficients, we multiply (*9*) on both sides by $e^{-jn\omega t}$ and integrate over the full period:

$$\int_0^{2\pi} f(t)e^{-jn\omega t}\,d(\omega t) = \cdots + \int_0^{2\pi} \mathbf{A}_{-2}e^{-j2\omega t}e^{-jn\omega t}\,d(\omega t) + \int_0^{2\pi} \mathbf{A}_{-1}e^{-j\omega t}e^{-jn\omega t}\,d(\omega t)$$
$$+ \int_0^{2\pi} \mathbf{A}_0 e^{-jn\omega t}\,d(\omega t) + \int_0^{2\pi} \mathbf{A}_1 e^{j\omega t}e^{-jn\omega t}\,d(\omega t) + \cdots$$
$$+ \int_0^{2\pi} \mathbf{A}_n e^{jn\omega t}e^{-jn\omega t}\,d(\omega t) + \cdots \tag{10}$$

The definite integrals on the right side of (*10*) are all zero except $\int_0^{2\pi} \mathbf{A}_n\,d(\omega t)$, which has the value $2\pi\mathbf{A}_n$. Then

$$\mathbf{A}_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-jn\omega t}\,d(\omega t) \qquad \text{or} \qquad \mathbf{A}_n = \frac{1}{T}\int_0^{T} f(t)e^{-j2\pi nt/T}\,dt \tag{11}$$

Just as with the a_n and b_n evaluation integrals, the limits of integration in (*11*) may be the endpoints of any convenient full period and not necessarily 0 to 2π or 0 to T. Note that, $f(t)$ being real, $\mathbf{A}_{-n} = \mathbf{A}_n^*$, so that only positive n needed to be considered in (*11*). Furthermore, we have

$$a_n = 2\,\mathrm{Re}\,\mathbf{A}_n \qquad b_n = -2\,\mathrm{Im}\,\mathbf{A}_n \tag{12}$$

EXAMPLE 17.2 Derive the exponential series (*9*) from the trigonometric series (*1*).

Replace the sine and cosine terms in (*1*) by their complex exponential equivalents.

$$\sin n\omega t = \frac{e^{jn\omega t} - e^{-jn\omega t}}{2j} \qquad \cos n\omega t = \frac{e^{jn\omega t} + e^{-jn\omega t}}{2}$$

Arranging the exponential terms in order of increasing n from $-\infty$ to $+\infty$, we obtain the infinite sum (*9*) where $A_0 = a_0/2$ and

$$\mathbf{A}_n = \tfrac{1}{2}(a_n - jb_n) \qquad \mathbf{A}_{-n} = \tfrac{1}{2}(a_n + jb_n) \qquad \text{for } n = 1, 2, 3, \ldots$$

EXAMPLE 17.3 Find the exponential Fourier series for the waveform shown in Fig. 17-2. Using the coefficients of this exponential series, obtain a_n and b_n of the trigonometric series and compare with Example 17.1.

In the interval $0 < \omega t < 2\pi$ the function is given by $f(t) = (10/2\pi)\omega t$. By inspection, the average value of the function is $A_0 = 5$. Substituting $f(t)$ in (*11*), we obtain the coefficients $\mathbf{A}_n$.

$$\mathbf{A}_n = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{10}{2\pi}\right)\omega t e^{-jn\omega t}\,d(\omega t) = \frac{10}{(2\pi)^2}\left[\frac{e^{-jn\omega t}}{(-jn)^2}(-jn\omega t - 1)\right]_0^{2\pi} = j\,\frac{10}{2\pi n}$$

Inserting the coefficients $\mathbf{A}_n$ in (*12*), the exponential form of the Fourier series for the given waveform is

$$f(t) = \cdots - j\,\frac{10}{4\pi}\,e^{-j2\omega t} - j\,\frac{10}{2\pi}\,e^{-j\omega t} + 5 + j\,\frac{10}{2\pi}\,e^{j\omega t} + j\,\frac{10}{4\pi}\,e^{j2\omega t} + \cdots \tag{13}$$

The trigonometric series coefficients are, by (*12*),

$$a_n = 0 \qquad b_n = -\frac{10}{\pi n}$$

and so
$$f(t) = 5 - \frac{10}{\pi}\sin\omega t - \frac{10}{2\pi}\sin 2\omega t - \frac{10}{3\pi}\sin 3\omega t - \cdots$$

which is the same as in Example 17.1.

17.4 WAVEFORM SYMMETRY

The series obtained in Example 17.1 contained only sine terms in addition to a constant term. Other waveforms will have only cosine terms; and sometimes only odd harmonics are present in the series, whether the series contains sine, cosine, or both types of terms. This is the result of certain types of symmetry exhibited by the waveform. Knowledge of such symmetry results in reduced calculations in determining the Fourier series. For this reason the following definitions are important.

1. A function $f(x)$ is said to be *even* if $f(x) = f(-x)$.

   The function $f(x) = 2 + x^2 + x^4$ is an example of even functions since the functional values for x and $-x$ are equal. The cosine is an even function, since it can be expressed as the power series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

The sum or product of two or more even functions is an even function, and with the addition of a constant the even nature of the function is still preserved.

In Fig. 17-3, the waveforms shown represent even functions of x. They are symmetrical with respect to the vertical axis, as indicated by the construction in Fig. 17-3(a).

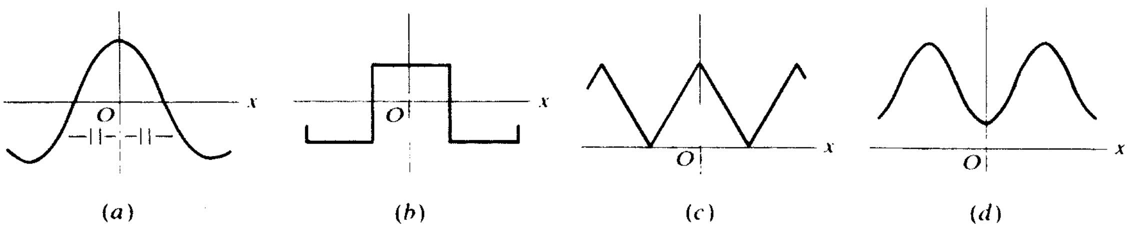


Fig. 17-3

2. A function $f(x)$ is said to be *odd* if $f(x) = -f(-x)$.

   The function $f(x) = x + x^3 + x^5$ is an example of odd functions since the values of the function for x and $-x$ are of opposite sign. The sine is an odd function, since it can be expressed as the power series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

The sum of two or more odd functions is an odd function, but the addition of a constant removes the odd nature of the function. The product of two odd functions is an even function.

The waveforms shown in Fig. 17-4 represent odd functions of x. They are symmetrical with respect to the origin, as indicated by the construction in Fig. 17-4(a).

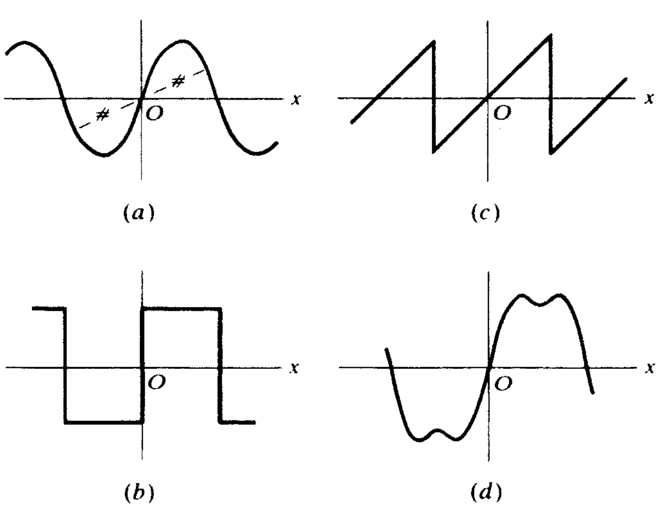


Fig. 17-4

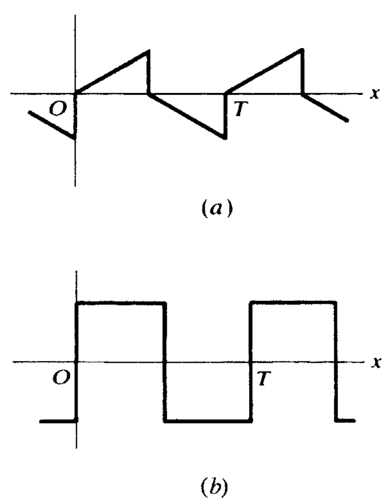


Fig. 17-5

3. A periodic function $f(x)$ is said to have *half-wave symmetry* if $f(x) = -f(x + T/2)$ where T is the period. Two waveforms with half-wave symmetry are shown in Fig. 17-5.

When the type of symmetry of a waveform is established, the following conclusions are reached. If the waveform is even, all terms of its Fourier series are cosine terms, including a constant if the waveform has a nonzero average value. Hence, there is no need of evaluating the integral for the coefficients b_n, since no sine terms can be present. If the waveform is odd, the series contains only sine terms. The wave may be odd only after its average value is subtracted, in which case its Fourier representation will simply contain that constant and a series of sine terms. If the waveform has half-wave symmetry, only odd harmonics are present in the series. This series will contain both sine and cosine terms unless the function is also odd or even. In any case, a_n and b_n are equal to zero for $n = 2, 4, 6, \ldots$ for any waveform with half-wave symmetry. Half-wave symmetry, too, may be present only after subtraction of the average value.

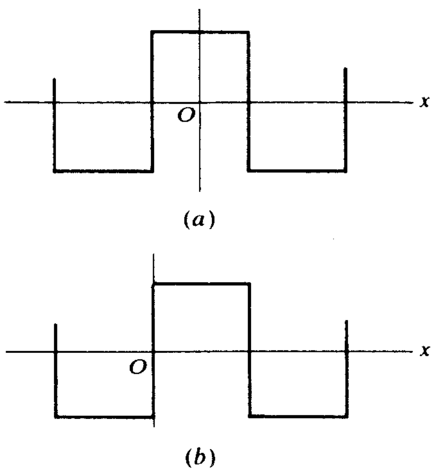


Fig. 17-6

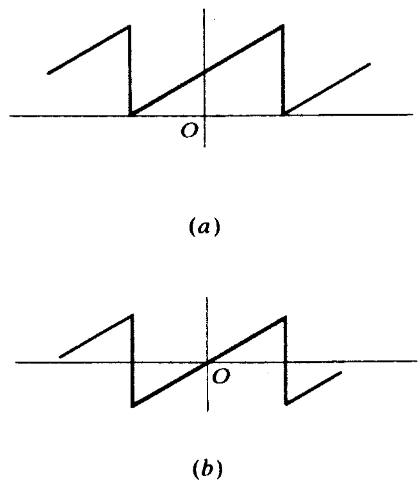


Fig. 17-7

Certain waveforms can be odd or even, depending upon the location of the vertical axis. The square wave of Fig. 17-6(*a*) meets the condition of an even function: $f(x) = f(-x)$. A shift of the vertical axis to the position shown in Fig. 17-6(*b*) produces an odd function $f(x) = -f(-x)$. With the vertical axis placed at any points other than those shown in Fig. 17-6, the square wave is neither even nor odd, and its series contains both sine and cosine terms. Thus, in the analysis of periodic functions, the vertical axis should be conveniently chosen to result in either an even or odd function, if the type of waveform makes this possible.

The shifting of the horizontal axis may simplify the series representation of the function. As an example, the waveform of Fig. 17-7(*a*) does not meet the requirements of an odd function until the average value is removed as shown in Fig. 17-7(*b*). Thus, its series will contain a constant term and only sine terms.

The preceding symmetry considerations can be used to check the coefficients of the exponential Fourier series. An even waveform contains only cosine terms in its trigonometric series, and therefore the exponential Fourier coefficients must be pure real numbers. Similarly, an odd function whose trigonometric series consists of sine terms has pure imaginary coefficients in its exponential series.

17.5 LINE SPECTRUM

A plot showing each of the harmonic amplitudes in the wave is called the *line spectrum*. The lines decrease rapidly for waves with rapidly convergent series. Waves with discontinuities, such as the sawtooth and square wave, have spectra with slowly decreasing amplitudes, since their series have strong high harmonics. Their 10th harmonics will often have amplitudes of significant value as compared to the fundamental. In contrast, the series for waveforms without discontinuities and with a generally smooth appearance will converge rapidly, and only a few terms are required to generate the wave. Such rapid convergence will be evident from the line spectrum where the harmonic amplitudes decrease rapidly, so that any above the 5th or 6th are insignificant.

The harmonic content and the line spectrum of a wave are part of the very nature of that wave and never change, regardless of the method of analysis. Shifting the origin gives the trigonometric series a completely different appearance, and the exponential series coefficients also change greatly. However, the same harmonics always appear in the series, and their amplitudes,

$$c_0 = |\tfrac{1}{2}a_0| \qquad \text{and} \qquad c_n = \sqrt{a_n^2 + b_n^2} \quad (n \geq 1) \tag{14}$$

or

$$c_0 = |A_0| \qquad \text{and} \qquad c_n = |\mathbf{A}_n| + |\mathbf{A}_{-n}| = 2|\mathbf{A}_n| \quad (n \geq 1) \tag{15}$$

remain the same. Note that when the exponential form is used, the amplitude of the nth harmonic combines the contributions of frequencies $+n\omega$ and $-n\omega$.

EXAMPLE 17.4 In Fig. 17-8, the sawtooth wave of Example 17.1 and its line spectrum are shown. Since there were only sine terms in the trigonometric series, the harmonic amplitudes are given directly by $\frac{1}{2}a_0$ and $|b_n|$. The same line spectrum is obtained from the exponential Fourier series, (*13*).

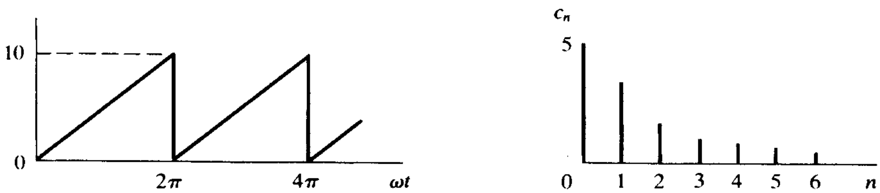


Fig. 17-8

17.6 WAVEFORM SYNTHESIS

Synthesis is a combination of parts so as to form a whole. Fourier synthesis is the recombination of the terms of the trigonometric series, usually the first four or five, to produce the original wave. Often it is only after synthesizing a wave that the student is convinced that the Fourier series does in fact represent the periodic wave for which it was obtained.

The trigonometric series for the sawtooth wave of Fig. 17-8 is

$$f(t) = 5 - \frac{10}{\pi}\sin\omega t - \frac{10}{2\pi}\sin 2\omega t - \frac{10}{3\pi}\sin 3\omega t - \cdots$$

These four terms are plotted and added in Fig. 17-9. Although the result is not a perfect sawtooth wave, it appears that with more terms included the sketch will more nearly resemble a sawtooth. Since this wave has discontinuities, its series is not rapidly convergent, and consequently, the synthesis using only four terms does not produce a very good result. The next term, at the frequency 4ω, has amplitude $10/4\pi$, which is certainly significant compared to the fundamental amplitude, $10/\pi$. As each term is added in the synthesis, the irregularities of the resultant are reduced and the approximation to the original wave is improved. This is what was meant when we said earlier that *the series converges to the function at all points of continuity and to the mean value at points of discontinuity*. In Fig. 17-9, at 0 and 2π it is clear that a value of 5 will remain, since all sine terms are zero at these points. These are the points of discontinuity; and the value of the function when they are approached from the left is 10, and from the right 0, with the mean value 5.

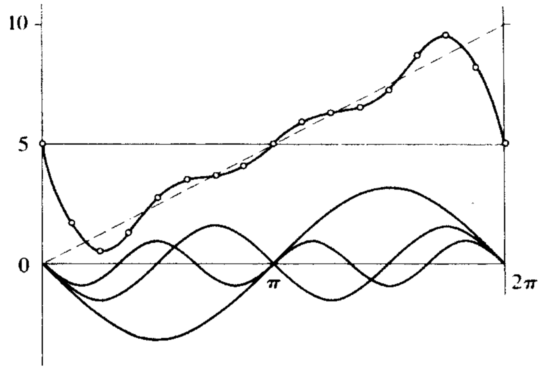


Fig. 17-9

17.7 EFFECTIVE VALUES AND POWER

The effective or rms value of the function

$$f(t) = \tfrac{1}{2}a_0 + a_1 \cos \omega t + a_2 \cos 2\omega t + \cdots + b_1 \sin \omega t + b_2 \sin 2\omega t + \cdots$$

is
$$F_{\text{rms}} = \sqrt{(\tfrac{1}{2}a_0)^2 + \tfrac{1}{2}a_1^2 + \tfrac{1}{2}a_2^2 + \cdots + \tfrac{1}{2}b_1^2 + \tfrac{1}{2}b_2^2 + \cdots} = \sqrt{c_0^2 + \tfrac{1}{2}c_1^2 + \tfrac{1}{2}c_2^2 + \tfrac{1}{2}c_3^3 + \cdots} \qquad (16)$$

where (*14*) has been used.

Considering a linear network with an applied voltage which is periodic, we would expect that the resulting current would contain the same harmonic terms as the voltage, but with harmonic amplitudes of different relative magnitude, since the impedance varies with $n\omega$. It is possible that some harmonics would not appear in the current; for example, in a pure *LC* parallel circuit, one of the harmonic frequencies might coincide with the resonant frequency, making the impedance at that frequency infinite. In general, we may write

$$v = V_0 + \sum V_n \sin(n\omega t + \phi_n) \qquad \text{and} \qquad i = I_0 + \sum I_n \sin(n\omega t + \psi_n) \qquad (17)$$

with corresponding effective values of

$$V_{\text{rms}} = \sqrt{V_0^2 + \tfrac{1}{2}V_1^2 + \tfrac{1}{2}V_2^2 + \cdots} \qquad \text{and} \qquad I_{\text{rms}} = \sqrt{I_0^2 + \tfrac{1}{2}I_1^2 + \tfrac{1}{2}I_2^2 + \cdots} \qquad (18)$$

The average power P follows from integration of the instantaneous power, which is given by the product of v and i:

$$p = vi = \Big[V_0 + \sum V_n \sin(n\omega t + \phi_n)\Big]\Big[I_0 + \sum I_n \sin(n\omega t + \psi_n)\Big] \qquad (19)$$

Since v and i both have period T, their product must have an integral number of its periods in T. (Recall that for a single sine wave of applied voltage, the product vi has a period half that of the voltage wave.) The average may therefore be calculated over one period of the voltage wave:

$$P = \frac{1}{T}\int_0^T \Big[V_0 + \sum V_n \sin(n\omega t + \phi_n)\Big]\Big[I_0 + \sum I_n \sin(n\omega t + \psi_n)\Big]\,dt \qquad (20)$$

Examination of the possible terms in the product of the two infinite series shows them to be of the following types: the product of two constants, the product of a constant and a sine function, the product of two sine functions of different frequencies, and sine functions squared. After integration, the product of the two constants is still $V_0 I_0$ and the sine functions squared with the limits applied appear as $(V_n I_n/2)\cos(\phi_n - \psi_n)$; all other products upon integration over the period T are zero. Then the average power is

$$P = V_0 I_0 + \tfrac{1}{2}V_1 I_1 \cos\theta_1 + \tfrac{1}{2}V_2 I_2 \cos\theta_2 + \tfrac{1}{2}V_3 I_3 \cos\theta_3 + \cdots \qquad (21)$$

where $\theta_n = \phi_n - \psi_n$ is the angle on the equivalent impedance of the network at the angular frequency $n\omega$, and V_n and I_n are the maximum values of the respective sine functions.

In the special case of a single-frequency sinusoidal voltage, $V_0 = V_2 = V_3 = \cdots = 0$, and (*21*) reduces to the familiar

$$P = \tfrac{1}{2} V_1 I_1 \cos\theta_1 = V_{\text{eff}} I_{\text{eff}} \cos\theta$$

Compare Section 10.2. Also, for a dc voltage, $V_1 = V_2 = V_3 = \cdots = 0$, and (*21*) becomes

$$P = V_0 I_0 = VI$$

Thus, (*21*) is quite general. Note that on the right-hand side there is no term that involves voltage and current of different frequencies. In regard to power, then, each harmonic acts independently, and

$$P = P_0 + P_1 + P_2 + \cdots$$

17.8 APPLICATIONS IN CIRCUIT ANALYSIS

It has already been suggested above that we could apply the terms of a voltage series to a linear network and obtain the corresponding harmonic terms of the current series. This result is obtained by superposition. Thus we consider each term of the Fourier series representing the voltage as a single source, as shown in Fig. 17.10. Now the equivalent impedance of the network at each harmonic frequency $n\omega$ is used to compute the current at that harmonic. The sum of these individual responses is the total response i, in series form, to the applied voltage.

EXAMPLE 17.5 A series RL circuit in which $R = 5\ \Omega$ and $L = 20$ mH (Fig. 17-11) has an applied voltage $v = 100 + 50 \sin\omega t + 25 \sin 3\omega t$ (V), with $\omega = 500$ rad/s. Find the current and the average power.

Compute the equivalent impedance of the circuit at each frequency found in the voltage function. Then obtain the respective currents.

At $\omega = 0$, $Z_0 = R = 5\ \Omega$ and

$$I_0 = \frac{V_0}{R} = \frac{100}{5} = 20 \text{ A}$$

At $\omega = 500$ rad/s, $\mathbf{Z}_1 = 5 + j(500)(20 \times 10^{-3}) = 5 + j10 = 11.15\underline{/63.4^\circ}\ \Omega$ and

$$i_1 = \frac{V_{1,\max}}{Z_1}\sin(\omega t - \theta_1) = \frac{50}{11.15}\sin(\omega t - 63.4^\circ) = 4.48 \sin(\omega t - 63.4^\circ) \quad \text{(A)}$$

At $3\omega = 1500$ rad/s, $\mathbf{Z}_3 = 5 + j30 = 30.4\underline{/80.54^\circ}\ \Omega$ and

$$i_3 = \frac{V_{3,\max}}{Z_3}\sin(3\omega t - \theta_3) = \frac{25}{30.4}\sin(3\omega t - 80.54^\circ) = 0.823 \sin(3\omega t - 80.54^\circ) \quad \text{(A)}$$

The sum of the harmonic currents is the required total response; it is a Fourier series of the type (*8*).

$$i = 20 + 4.48 \sin(\omega t - 63.4^\circ) + 0.823 \sin(3\omega t - 80.54^\circ) \quad \text{(A)}$$

This current has the effective value

$$I_{\text{eff}} = \sqrt{20^2 + (4.48^2/2) + (0.823^2/2)} = \sqrt{410.6} = 20.25 \text{ A}$$

which results in a power in the 5-Ω resistor of

$$P = I_{\text{eff}}^2 R = (410.6)5 = 2053 \text{ W}$$

As a check, we compute the total average power by calculating first the power contributed by each harmonic and then adding the results.

At $\omega = 0$:	$P_0 = V_0 I_0 = 100(20) = 2000$ W
At $\omega = 500$ rad/s:	$P_1 = \frac{1}{2} V_1 I_1 \cos\theta_1 = \frac{1}{2}(50)(4.48)\cos 63.4^\circ = 50.1$ W
At $3\omega = 1500$ rad/s:	$P_3 = \frac{1}{2} V_3 I_3 \cos\theta_3 = \frac{1}{2}(25)(0.823)\cos 80.54^\circ = 1.69$ W
Then,	$P = 2000 + 50.1 + 1.69 = 2052$ W

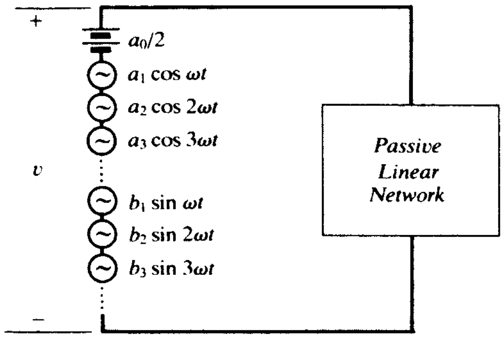


Fig. 17-10

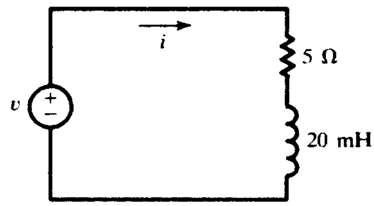


Fig. 17-11

Another Method

The Fourier series expression for the voltage across the resistor is

$$v_R = Ri = 100 + 22.4\sin(\omega t - 63.4^\circ) + 4.11\sin(3\omega t - 80.54^\circ) \quad \text{(V)}$$

and

$$V_{R\text{eff}} = \sqrt{100^2 + \frac{1}{2}(22.4)^2 + \frac{1}{2}(4.11)^2} = \sqrt{10\,259} = 101.3 \text{ V}$$

Then the power delivered by the source is $P = V_{R\text{eff}}^2/R = (10\,259)/5 = 2052$ W.

In Example 17.5 the driving voltage was given as a trigonometric Fourier series in t, and the computations were in the time domain. (The complex impedance was used only as a shortcut; Z_n and θ_n could have been obtained directly from R, L, and $n\omega$). If, instead, the voltage is represented by an exponential Fourier series,

$$v(t) = \sum_{-\infty}^{+\infty} \mathbf{V}_n e^{jn\omega t}$$

then we have to do with a superposition of *phasors* $\mathbf{V}_n$ (rotating counterclockwise if $n > 0$, clockwise if $n < 0$), and so frequency-domain methods are called for. This is illustrated in Example 17.6.

EXAMPLE 17.6 A voltage represented by the triangular wave shown in Fig. 17-12 is applied to a pure capacitor C. Determine the resulting current.

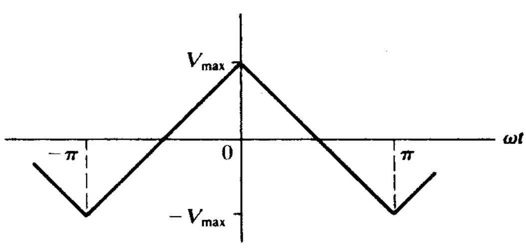


Fig. 17-12

In the interval $-\pi < \omega t < 0$ the voltage function is $v = V_{\max} + (27V_{\max}/\pi)\omega t$; and for $0 < \omega t < \pi$, $v = V_{\max} - (2V_{\max}/\pi)\omega t$. Then the coefficients of the exponential series are determined by the evaluation integral

$$\mathbf{V}_n = \frac{1}{2\pi}\int_{-\pi}^{0} [V_{\max} + (2V_{\max}/\pi)\omega t]e^{-jn\omega t}\, d(\omega t) + \frac{1}{2\pi}\int_{0}^{\pi} [V_{\max} - (2V_{\max}/\pi)\omega t]e^{-jn\omega t}\, d(\omega t)$$

from which $\mathbf{V}_n = 4V_{\max}/\pi^2 n^2$ for odd n, and $\mathbf{V}_n = 0$ for even n.

The phasor current produced by $\mathbf{V}_n$ (n odd) is

$$\mathbf{I}_n = \frac{\mathbf{V}_n}{\mathbf{Z}_n} = \frac{4V_{\max}/\pi^2 n^2}{1/jn\omega C} = j\,\frac{4V_{\max}\omega C}{\pi^2 n}$$

with an implicit time factor $e^{jn\omega t}$. The resultant current is therefore

$$i(t) = \sum_{-\infty}^{+\infty} \mathbf{I}_n e^{jn\omega t} = j\,\frac{4V_{\max}\omega C}{\pi^2} \sum_{-\infty}^{+\infty} \frac{e^{jn\omega t}}{n}$$

where the summation is over odd n only.

The series could be converted to the trigonometric form and then synthesized to show the current waveform. However, this series is of the same form as the result in Problem 17.8, where the coefficients are $\mathbf{A}_n = -j(2V/n\pi)$ for odd n only. The sign here is negative, indicating that our current wave is the negative of the square wave of Problem 17.8 and has a peak value $2V_{\max}\omega C/\pi$.

17.9 FOURIER TRANSFORM OF NONPERIODIC WAVEFORMS

A nonperiod waveform $x(t)$ is said to satisfy the Dirichlet conditions if

(*a*) $x(t)$ is absolutely integrable, $\int_{-\infty}^{+\infty} |x(t)|\,dt < \infty$, and

(*b*) the number of maxima and minima and the number of discontinuities of $x(t)$ in every finite interval is finite.

For such a waveform, we can define the Fourier transform $\mathbf{X}(f)$ by

$$\mathbf{X}(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt \qquad (22a)$$

where f is the frequency. The above integral is called the *Fourier integral*. The time function $x(t)$ is called the *inverse Fourier transform* of $\mathbf{X}(f)$ and is obtained from it by

$$x(t) = \int_{-\infty}^{\infty} \mathbf{X}(f)e^{j2\pi ft}\,df \qquad (22b)$$

$x(t)$ and $\mathbf{X}(f)$ form a Fourier transform pair. Instead of f, the angular velocity $\omega = 2\pi f$ may also be used, in which case, (*22a*) and (*22b*) become, respectively,

$$\mathbf{X}(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \qquad (23a)$$

and

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega \qquad (23b)$$

EXAMPLE 17.7 Find the Fourier transform of $x(t) = e^{-at}u(t)$, $a > 0$. Plot $\mathbf{X}(f)$ for $-\infty < f < +\infty$.

From (*22a*), the Fourier transform of $x(t)$ is

$$\mathbf{X}(f) = \int_{0}^{\infty} e^{-at}e^{-j2\pi ft}\,dt = \frac{1}{a + j2\pi f} \qquad (24)$$

$\mathbf{X}(f)$ is a complex function of a real variable. Its magnitude and phase angle, $|\mathbf{X}(f)|$ and $\underline{/\mathbf{X}(f)}$, respectively, shown in Figs. 17-13(*a*) and (*b*), are given by

$$|\mathbf{X}(f)| = \frac{1}{\sqrt{a^2 + 4\pi^2 f^2}} \qquad (25a)$$

and

$$\underline{/\mathbf{X}(f)} = -\tan^{-1}(2\pi f/a) \qquad (25b)$$

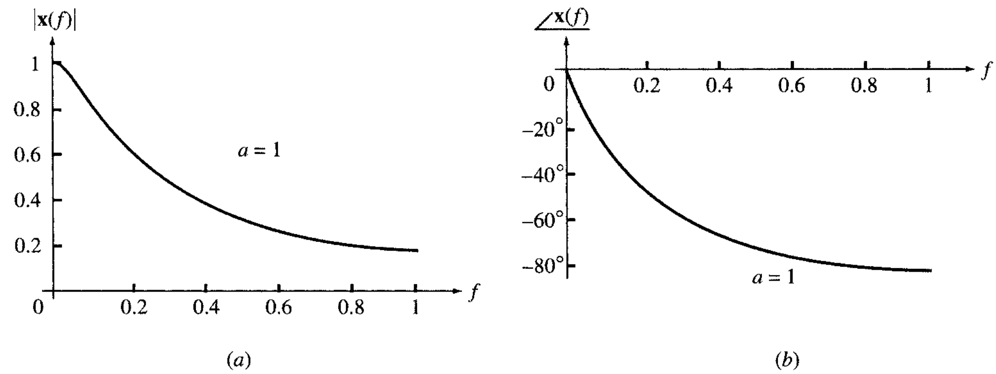


Fig. 17-13

Alternatively, $\mathbf{X}(f)$ may be shown by its real and imaginary parts, Re $[\mathbf{X}(f)]$ and Im $[\mathbf{X}(f)]$, as in Figs. 17-14(*a*) and (*b*).

$$\mathrm{Re}\,[\mathbf{X}(f)] = \frac{a}{a^2 + 4\pi^2 f^2} \tag{26a}$$

$$\mathrm{Im}\,[\mathbf{X}(f)] = \frac{-2\pi f}{a^2 + 4\pi^2 f^2} \tag{26b}$$

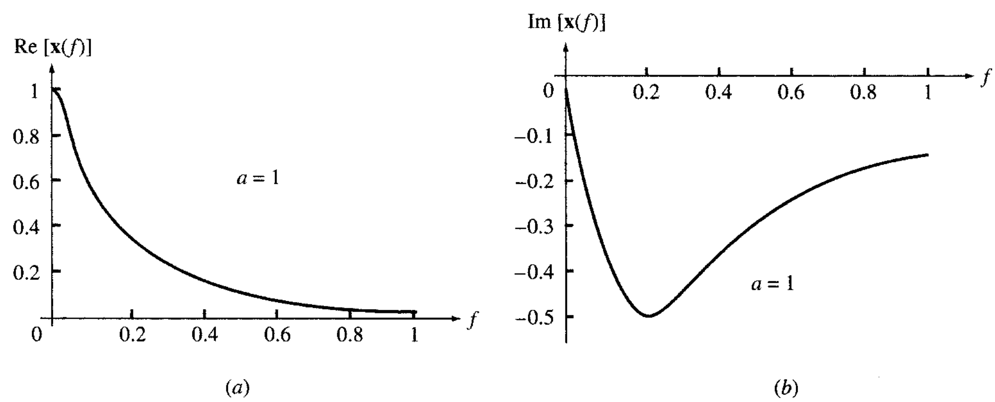


Fig. 17-14

EXAMPLE 17.8 Find the Fourier transform of the square pulse

$$x(t) = \begin{cases} 1 & \text{for } -T < t < T \\ 0 & \text{otherwise} \end{cases}$$

From (*22a*),

$$\mathbf{X}(f) = \int_{-T}^{T} e^{-j2\pi ft}\,dt = \frac{1}{-j2\pi f}\left[e^{j2\pi f}\right]_{-T}^{T} = \frac{\sin 2\pi fT}{\pi f} \tag{27}$$

Because $x(t)$ is even, $\mathbf{X}(f)$ is real. The transform pairs are plotted in Figs. 17-15(*a*) and (*b*) for $T = \frac{1}{2}$ s.

EXAMPLE 17.9 Find the Fourier transform of $x(t) = e^{at}u(-t)$, $a > 0$.

$$\mathbf{X}(f) = \int_{-\infty}^{0} e^{at} e^{-j2\pi ft}\,dt = \frac{1}{a - j2\pi f} \tag{28}$$

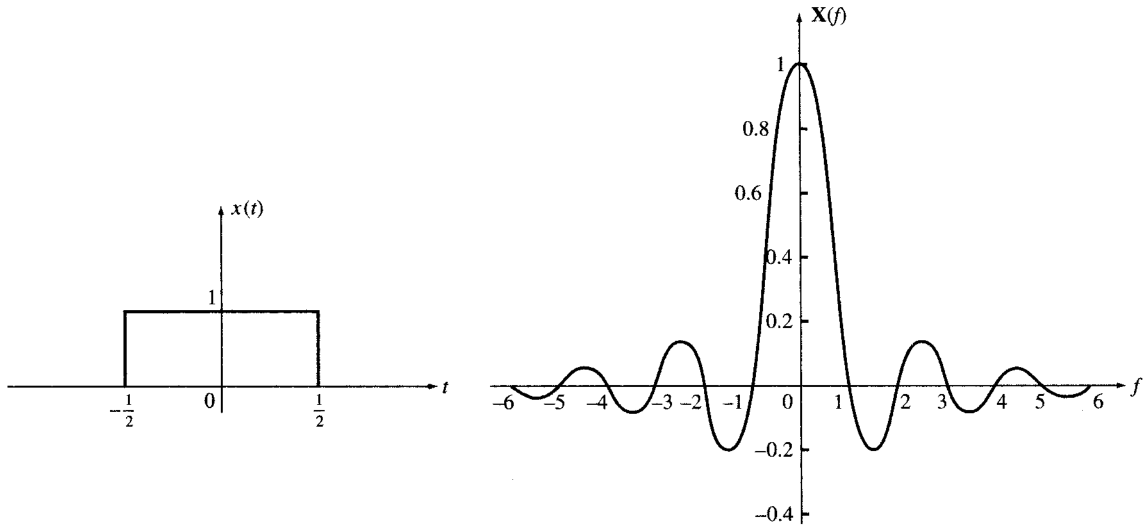


Fig. 17-15

EXAMPLE 17.10 Find the inverse Fourier transform of $\mathbf{X}(f) = 2a/(a^2 + 4\pi^2 f^2)$, $a > 0$.
By partial fraction expansion we have

$$\mathbf{X}(f) = \frac{1}{a + j2\pi f} + \frac{1}{a - j2\pi f} \tag{29}$$

The inverse of each term in (*29*) may be derived from (*24*) and (*28*) so that

$$x(t) = e^{-at}u(t) + e^{at}u(-t) = e^{-a|t|} \qquad \text{for all } t$$

See Fig. 17-16.

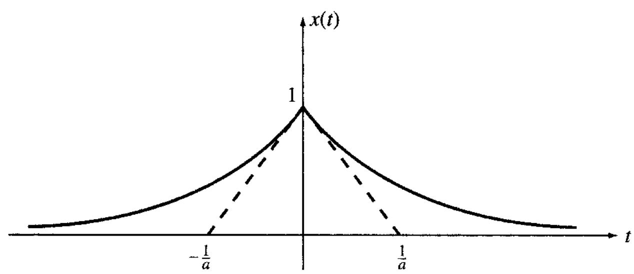


Fig. 17-16

17.10 PROPERTIES OF THE FOURIER TRANSFORM

Some properties of the Fourier transform are listed in Table 17-1. Several commonly used transform pairs are given in Table 17-2.

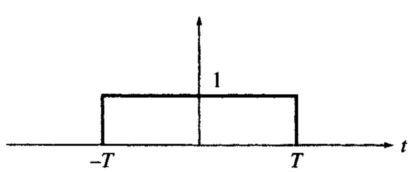
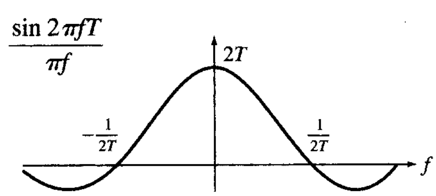
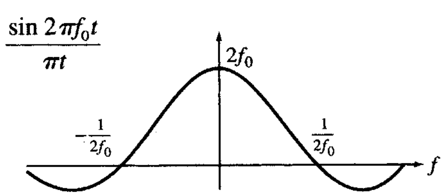
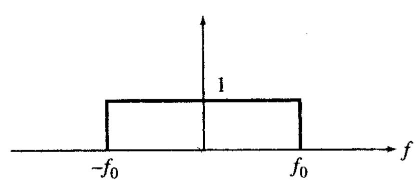
17.11 CONTINUOUS SPECTRUM

$|\mathbf{X}(f)|^2$, as defined in Section 17.9, is called the *energy density* or the *spectrum* of the waveform $x(t)$. Unlike the periodic functions, the energy content of a nonperiodic waveform $x(t)$ at each frequency is zero. However, the energy content within a frequency band from f_1 to f_2 is

Table 17-1 Fourier Transform Properties

	Time Domain $x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\,dt$	Frequency Domain $\mathbf{X}(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt$
1.	$x(t)$ real	$X(f) = X^*(-f)$
2.	$x(t)$ even, $x(t) = x(-t)$	$X(f) = X(-f)$
3.	$x(t)$, odd, $x(t) = -x(-t)$	$X(f) = -X(-f)$
4.	$X(t)$	$x(-f)$
5.	$x(0) = \int_{-\infty}^{\infty} X(f)\,df$	$X(0) = \int_{-\infty}^{\infty} x(t)\,dt$
6.	$y(t) = x(at)$	$Y(f) = \dfrac{1}{\lvert a\rvert} X(f/a)$
7.	$y(t) = tx(t)$	$Y(f) = -\dfrac{1}{j2\pi}\dfrac{dX(f)}{df}$
8.	$y(t) = x(-t)$	$Y(f) = X(-f)$
9.	$y(t) = x(t - t_0)$	$Y(f) = e^{-j2\pi f t_0} X(f)$

Table 17-2 Fourier Transform Pairs

	$x(t)$	$X(f)$
1.	$e^{-at}u(t),\ a > 0$	$\dfrac{1}{a + j2\pi f}$
2.	$e^{-a\lvert t\rvert},\ a > 0$	$\dfrac{2a}{a^2 + 4\pi^2 f^2}$
3.	$te^{-at}u(t),\ a > 0$	$\dfrac{1}{(a + j2\pi f)^2}$
4.	$\exp(-\pi t^2/\tau^2)$	$\tau \exp(-\pi f^2 \tau^2)$
5.		
6.		
7.	1	$\delta(f)$
8.	$\delta(t)$	1
9.	$\sin 2\pi f_0 t$	$\dfrac{\delta(f - f_0) - \delta(f + f_0)}{2j}$
10.	$\cos 2\pi f_0 t$	$\dfrac{\delta(f - f_0) + \delta(f + f_0)}{2}$

$$W = 2\int_{f_1}^{f_2} |\mathbf{x}(f)|^2\, df \tag{30}$$

EXAMPLE 17.11 Find the spectrum of $x(t) = e^{-at}u(t) - e^{at}u(-t)$, $a > 0$, shown in Fig. 17-17.

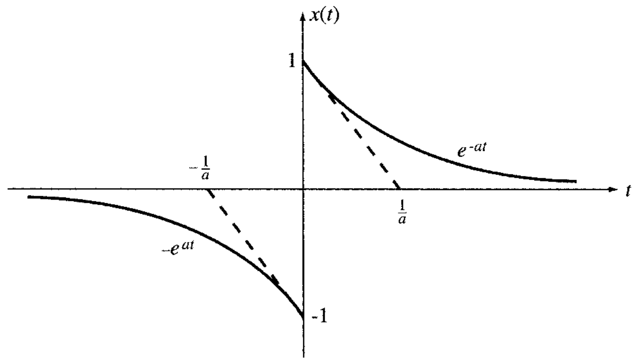


Fig. 17-17

We have $x(t) = x_1(t) - x_2(t)$. Since $x_1(t) = e^{-at}u(t)$ and $x_2(t) = e^{at}u(-t)$,

$$\mathbf{X}_1(f) = \frac{1}{a + j2\pi f} \qquad \mathbf{X}_2(f) = \frac{1}{a - j2\pi f}$$

Then

$$\mathbf{X}(f) = \mathbf{X}_1(f) - \mathbf{X}_2(f) = \frac{-j4\pi f}{a^2 + 4\pi^2 f^2}$$

from which

$$|\mathbf{X}(f)|^2 = \frac{16\pi^2 f^2}{(a^2 + 4\pi^2 f^2)^2}$$

EXAMPLE 17.12 Find and compare the energy contents W_1 and W_2 of $y_1(t) = e^{-|at|}$ and $y_2(t) = e^{-at}u(t) - e^{at}u(-t)$, $a > 0$, within the band 0 to 1 Hz. Let $a = 200$.

From Examples 17.10 and 17.11,

$$|\mathbf{Y}_1(f)|^2 = \frac{4a^2}{(a^2 + 4\pi^2 f^2)^2} \qquad \text{and} \qquad |\mathbf{Y}_2(f)|^2 = \frac{16\pi^2 f^2}{(a^2 + 4\pi^2 f^2)^2}$$

Within $0 < f < 1$ Hz, the spectra and energies may be approximated by

$$|\mathbf{Y}_1(f)|^2 \approx 4/a^2 = 10^{-4}\ \text{J/Hz} \qquad \text{and} \qquad W_1 = 2(10^{-4})\ \text{J} = 200\ \mu\text{J}$$
$$|\mathbf{Y}_2(f)^2| \approx 10^{-7} f^2 \qquad \text{and} \qquad W_2 \approx 0$$

The preceding results agree with the observation that most of the energy in $y_1(t)$ is near the low-frequency region in contrast to $y_2(t)$.

Solved Problems

17.1 Find the trigonometric Fourier series for the square wave shown in Fig. 17-18 and plot the line spectrum.

In the interval $0 < \omega t < \pi, f(t) = V$; and for $\pi < \omega t < 2\pi, f(t) = -V$. The average value of the wave is zero; hence, $a_0/2 = 0$. The cosine coefficients are obtained by writing the evaluation integral with the functions inserted as follows:

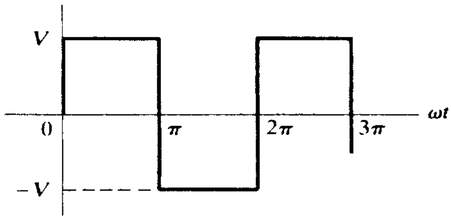


Fig. 17-18

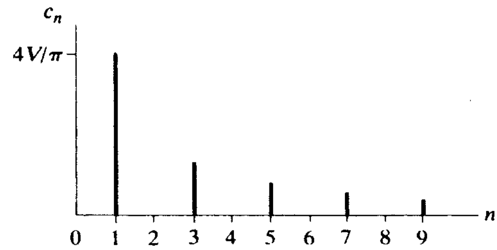


Fig. 17-19

$$a_n = \frac{1}{\pi}\left[\int_0^{\pi} V\cos n\omega t\, d(\omega t) + \int_{\pi}^{2\pi}(-V)\cos n\omega t\, d(\omega t)\right] = \frac{V}{\pi}\left\{\left[\frac{1}{n}\sin n\omega t\right]_0^{\pi} - \left[\frac{1}{n}\sin n\omega t\right]_{\pi}^{2\pi}\right\}$$
$$= 0 \qquad \text{for all } n$$

Thus, the series contains no cosine terms. Proceeding with the evaluation integral for the sine terms,

$$b_n = \frac{1}{\pi}\left[\int_0^{\pi} V\sin n\omega t\, d(\omega t) + \int_{\pi}^{2\pi}(-V)\sin n\omega t\, d(\omega t)\right]$$
$$= \frac{V}{\pi}\left\{\left[-\frac{1}{n}\cos n\omega t\right]_0^{\pi} + \left[\frac{1}{n}\cos n\omega t\right]_{\pi}^{2\pi}\right\}$$
$$= \frac{V}{\pi n}(-\cos n\pi + \cos 0 + \cos n2\pi - \cos n\pi) = \frac{2V}{\pi n}(1 - \cos n\pi)$$

Then $b_n = 4V/\pi n$ for $n = 1, 3, 5, \ldots$, and $b_n = 0$ for $n = 2, 4, 6, \ldots$. The series for the square wave is

$$f(t) = \frac{4V}{\pi}\sin\omega t + \frac{4V}{3\pi}\sin 3\omega t + \frac{4V}{5\pi}\sin 5\omega t + \cdots$$

The line spectrum for this series is shown in Fig. 17-19. This series contains only odd-harmonic sine terms, as could have been anticipated by examination of the waveform for symmetry. Since the wave in Fig. 17-18 is odd, its series contains only sine terms; and since it also has half-wave symmetry, only odd harmonics are present.

17.2 Find the trigonometric Fourier series for the triangular wave shown in Fig. 17-20 and plot the line spectrum.

The wave is an even function, since $f(t) = f(-t)$, and if its average value, $V/2$, is subtracted, it also has half-wave symmetry, that is, $f(t) = -f(t+\pi)$. For $-\pi < \omega t < 0$, $f(t) = V + (V/\pi)\omega t$; and for $0 < \omega t < \pi$, $f(t) = V - (V/\pi)\omega t$. Since even waveforms have only cosine terms, all $b_n = 0$. For $n \geq 1$,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0}[V + (V/\pi)\omega t]\cos n\omega t\, d(\omega t) + \frac{1}{\pi}\int_0^{\pi}[V - (V/\pi)\omega t]\cos n\omega t\, d(\omega t)$$
$$= \frac{V}{\pi}\left[\int_{-\pi}^{\pi}\cos n\omega t\, d(\omega t) + \int_{-\pi}^{0}\frac{\omega t}{\pi}\cos n\omega t\, d(\omega t) - \int_0^{\pi}\frac{\omega t}{\pi}\cos n\omega t\, d(\omega t)\right]$$
$$= \frac{V}{\pi^2}\left\{\left[\frac{1}{n^2}\cos n\omega t + \frac{\omega t}{\pi}\sin n\omega t\right]_{-\pi}^{0} - \left[\frac{1}{n^2}\cos n\omega t + \frac{\omega t}{n}\sin n\omega t\right]_0^{\pi}\right\}$$
$$= \frac{V}{\pi^2 n^2}[\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0] = \frac{2V}{\pi^2 n^2}(1 - \cos n\pi)$$

As predicted from half-wave symmetry, the series contains only odd terms, since $a_n = 0$ for $n = 2, 4, 6, \ldots$. For $n = 1, 3, 5, \ldots$, $a_n = 4V/\pi^2 n^2$. Then the required Fourier series is

$$f(t) = \frac{V}{2} + \frac{4V}{-\pi^2}\cos\omega t + \frac{4V}{(3\pi)^2}\cos 3\omega t + \frac{4V}{(5\pi)^2}\cos 5\omega t + \cdots$$

The coefficients decrease as $1/n^2$, and thus the series converges more rapidly than that of Problem 17.1. This fact is evident from the line spectrum shown in Fig. 17-21.

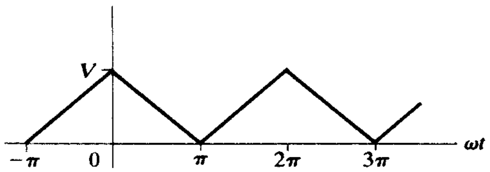


Fig. 17-20

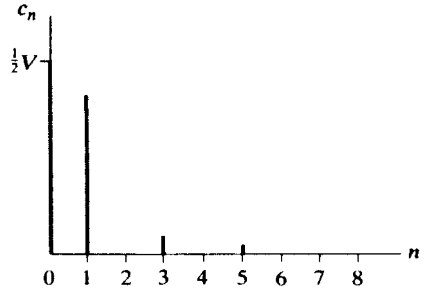


Fig. 17-21

17.3 Find the trigonometric Fourier series for the sawtooth wave shown in Fig. 17-22 and plot the line spectrum.

By inspection, the waveform is odd (and therefore has average value zero). Consequently the series will contain only sine terms. A single expression, $f(t) = (V/\pi)\omega t$, describes the wave over the period from $-\pi$ to $+\pi$, and we will use these limits on our evaluation integral for b_n.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (V/\pi)\omega t \sin n\omega t \, d(\omega t) = \frac{V}{\pi^2}\left[\frac{1}{n^2}\sin n\omega t - \frac{\omega t}{n}\cos n\omega t\right]_{-\pi}^{\pi} = -\frac{2V}{n\pi}(\cos n\pi)$$

As $\cos n\pi$ is $+1$ for even n and -1 for odd n, the signs of the coefficients alternate. The required series is

$$f(t) = \frac{2V}{\pi}\{\sin \omega t - \tfrac{1}{2}\sin 2\omega t + \tfrac{1}{3}\sin 3\omega t - \tfrac{1}{4}\sin 4\omega t + \cdots\}$$

The coefficients decrease as $1/n$, and thus the series converges slowly, as shown by the spectrum in Fig. 17-23. Except for the shift in the origin and the average term, this waveform is the same as in Fig. 17-8; compare the two spectra.

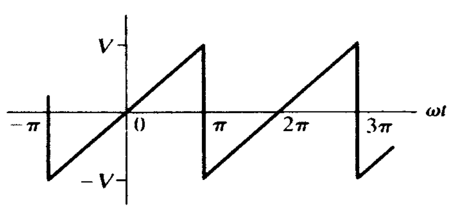


Fig. 17-22

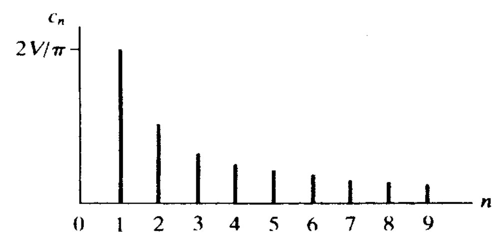


Fig. 17-23

17.4 Find the trigonometric Fourier series for the waveform shown in Fig. 17-24 and sketch the line spectrum.

In the interval $0 < \omega t < \pi$, $f(t) = (V/\pi)\omega t$; and for $\pi < \omega t < 2\pi$, $f(t) = 0$. By inspection, the average value of the wave is $V/4$. Since the wave is neither even nor odd, the series will contain both sine and cosine terms. For $n > 0$, we have

$$a_n = \frac{1}{\pi}\int_0^{\pi} (V/\pi)\omega t \cos n\omega t \, d(\omega t) = \frac{V}{\pi^2}\left[\frac{1}{n^2}\cos n\omega t + \frac{\omega t}{n}\sin n\omega t\right]_0^{\pi} = \frac{V}{\pi^2 n^2}(\cos n\pi - 1)$$

When n is even, $\cos n\pi - 1 = 0$ and $a_n = 0$. When n is odd, $a_n = -2V/(\pi^2 n^2)$. The b_n coefficients are

$$b_n = \frac{1}{\pi}\int_0^{\pi} (V/\pi)\omega t \sin n\omega t \, d(\omega t) = \frac{V}{\pi^2}\left[\frac{1}{n^2}\sin n\omega t - \frac{\omega t}{n}\cos n\omega t\right]_0^{\pi} = -\frac{V}{\pi n}(\cos n\pi) = (-1)^{n+1}\frac{V}{\pi n}$$

Then the required Fourier series is

$$f(t) = \frac{V}{4} - \frac{2V}{\pi^2}\cos\omega t - \frac{2V}{(3\pi)^2}\cos 3\omega t - \frac{2V}{(5\pi)^2}\cos 5\omega t - \cdots$$
$$+ \frac{V}{\pi}\sin\omega t - \frac{V}{2\pi}\sin 2\omega t + \frac{V}{3\pi}\sin 3\omega t - \cdots$$

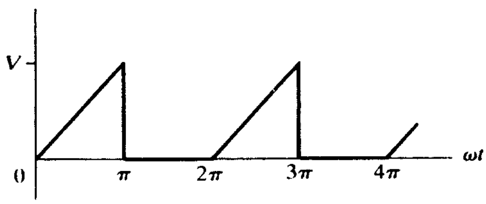


Fig. 17-24

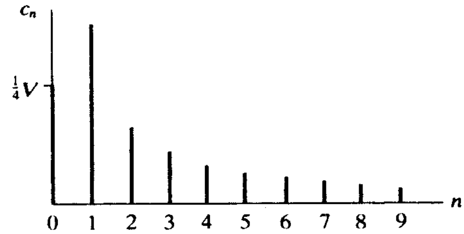


Fig. 17-25

The even-harmonic amplitudes are given directly by $|b_n|$, since there are no even-harmonic cosine terms. However, the odd-harmonic amplitudes must be computed using $c_n = \sqrt{a_n^2 + b_n^2}$. Thus,

$$c_1 = \sqrt{(2V/\pi^2)^2 + (V/\pi)^2} = V(0.377) \qquad c_3 = V(0.109) \qquad c_5 = V(0.064)$$

The line spectrum is shown in Fig. 17-25.

17.5 Find the trigonometric Fourier series for the half-wave-rectified sine wave shown in Fig. 17-26 and sketch the line spectrum.

The wave shows no symmetry, and we therefore expect the series to contain both sine and cosine terms. Since the average value is not obtainable by inspection, we evaluate a_0 for use in the term $a_0/2$.

$$a_0 = \frac{1}{\pi}\int_0^{\pi} V \sin\omega t \, d(\omega t) = \frac{V}{\pi}[-\cos\omega t]_0^{\pi} = \frac{2V}{\pi}$$

Next we determine a_n:

$$a_n = \frac{1}{\pi}\int_0^{\pi} V \sin\omega t \cos n\omega t \, d(\omega t)$$
$$= \frac{V}{\pi}\left[\frac{-n\sin\omega t \sin n\omega t - \cos n\omega t \cos\omega t}{-n^2+1}\right]_0^{\pi} = \frac{V}{\pi(1-n^2)}(\cos n\pi + 1)$$

With n even, $a_n = 2V/\pi(1-n^2)$; and with n odd, $a_n = 0$. However, this expression is indeterminate for $n = 1$, and therefore we must integrate separately for a_1.

$$a_1 = \frac{1}{\pi}\int_0^{\pi} V \sin\omega t \cos\omega t \, d(\omega t) = \frac{V}{\pi}\int_0^{\pi} \tfrac{1}{2}\sin 2\omega t \, d(\omega t) = 0$$

Now we evaluate b_n:

$$b_n = \frac{1}{\pi}\int_0^{\pi} V \sin\omega t \sin n\omega t \, d(\omega t) = \frac{V}{\pi}\left[\frac{n\sin\omega t \cos n\omega t - \sin n\omega t \cos\omega t}{-n^2+1}\right]_0^{\pi} = 0$$

Here again the expression is indeterminate for $n = 1$, and b_1 is evaluated separately.

$$b_1 = \frac{1}{\pi}\int_0^{\pi} V \sin^2 \omega t \, d(\omega t) = \frac{V}{\pi}\left[\frac{\omega t}{2} - \frac{\sin 2\omega t}{4}\right]_0^{\pi} = \frac{V}{2}$$

Then the required Fourier series is

$$f(t) = \frac{V}{\pi}\left(1 + \frac{\pi}{2}\sin \omega t - \frac{2}{3}\cos 2\omega t - \frac{2}{15}\cos 4\omega t - \frac{2}{35}\cos 6\omega t - \cdots\right)$$

The spectrum, Fig. 17-27, shows the strong fundamental term in the series and the rapidly decreasing amplitudes of the higher harmonics.

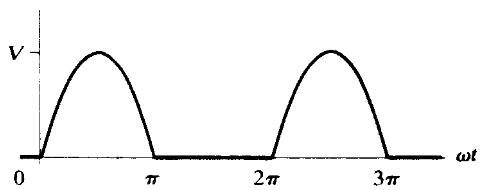


Fig. 17-26

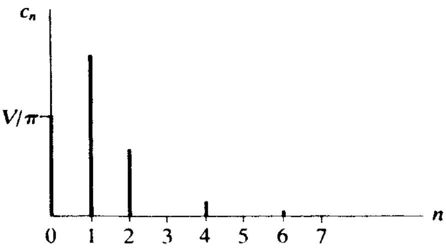


Fig. 17-27

17.6 Find the trigonometric Fourier series for the half-wave-rectified sine wave shown in Fig. 17-28, where the vertical axis is shifted from its position in Fig. 17-26.

The function is described in the interval $-\pi < \omega t < 0$ by $f(t) = -V \sin \omega t$. The average value is the same as that in Problem 17.5, that is, $\frac{1}{2}a_0 = V/\pi$. For the coefficients a_n, we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} (-V \sin \omega t)\cos n\omega t \, d(\omega t) = \frac{V}{\pi(1 - n^2)}(1 + \cos n\pi)$$

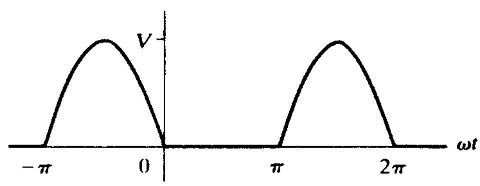


Fig. 17-28

For n even, $a_n = 2V/\pi(1 - n^2)$; and for n odd, $a_n = 0$, except that $n = 1$ must be examined separately.

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{0} (-V \sin \omega t)\cos \omega t \, d(\omega t) = 0$$

For the coefficients b_n, we obtain

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} (-V \sin \omega t)\sin n\omega t \, d(\omega t) = 0$$

except for $n = 1$.

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{0} (-V)\sin^2 \omega t \, d(\omega t) = -\frac{V}{2}$$

Thus, the series is

$$f(t) = \frac{V}{\pi}\left(1 - \frac{\pi}{2}\sin \omega t - \frac{2}{3}\cos 2\omega t - \frac{2}{15}\cos 4\omega t - \frac{2}{35}\cos 6\omega t - \cdots\right)$$

This series is identical to that of Problem 17.5, except for the fundamental term, which has a negative coefficient in this series. The spectrum would obviously be identical to that of Fig. 17-27.

Another Method

When the sine wave $V \sin \omega t$ is subtracted from the graph of Fig. 17.26, the graph of Fig. 17-28 results.

17.7 Obtain the trigonometric Fourier series for the repeating rectangular pulse shown in Fig. 17-29 and plot the line spectrum.

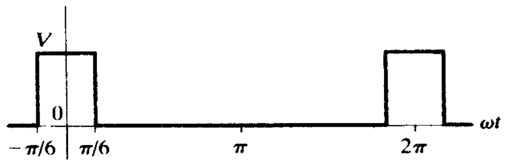


Fig. 17-29

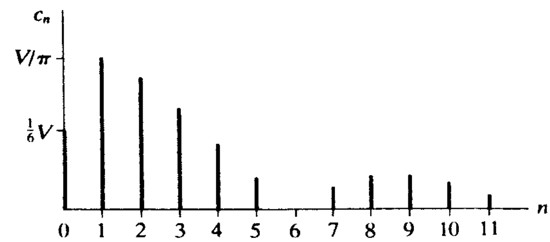


Fig. 17-30

With the vertical axis positioned as shown, the wave is even and the series will contain only cosine terms and a constant term. In the period from $-\pi$ to $+\pi$ used for the evaluation integrals, the function is zero except from $-\pi/6$ to $+\pi/6$.

$$a_0 = \frac{1}{\pi}\int_{-\pi/6}^{\pi/6} V\, d(\omega t) = \frac{V}{3} \qquad a_n = \frac{1}{\pi}\int_{-\pi/6}^{\pi/6} V \cos n\omega t\, d(\omega t) = \frac{2V}{n\pi} \sin \frac{n\pi}{6}$$

Since $\sin n\pi/6 = 1/2, \sqrt{3}/2, 1, \sqrt{3}/2, 1/2, 0, -1/2, \ldots$ for $n = 1, 2, 3, 4, 5, 6, 7, \ldots$, respectively, the series is

$$f(t) = \frac{V}{6} + \frac{2V}{\pi}\left[\frac{1}{2}\cos \omega t + \frac{\sqrt{3}}{2}\left(\frac{1}{2}\right)\cos 2\omega t + 1\left(\frac{1}{3}\right)\cos 3\omega t + \frac{\sqrt{3}}{2}\left(\frac{1}{4}\right)\cos 4\omega t + \frac{1}{2}\left(\frac{1}{5}\right)\cos 5\omega t - \frac{1}{2}\left(\frac{1}{7}\right)\cos 7\omega t - \cdots\right]$$

or

$$f(t) = \frac{V}{6} + \frac{2V}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi/6)\cos n\omega t$$

The line spectrum, shown in Fig. 17-30, decreases very slowly for this wave, since the series converges very slowly to the function. Of particular interest is the fact that the 8th, 9th, and 10th harmonic amplitudes exceed the 7th. With the simple waves considered previously, the higher-harmonic amplitudes were progressively lower.

17.8 Find the exponential Fourier series for the square wave shown in Figs. 17-18 and 17-31, and sketch the line spectrum. Obtain the trigonometric series coefficients from those of the exponential series and compare with Problem 17.1.

In the interval $-\pi < \omega t < 0$, $f(t) = -V$; and for $0 < \omega t < \pi$, $f(t) = V$. The wave is odd; therefore, $A_0 = 0$ and the $\mathbf{A}_n$ will be pure imaginaries.

$$\begin{aligned}\mathbf{A}_n &= \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-V)e^{-jn\omega t}\, d(\omega t) + \int_0^{\pi} Ve^{-jn\omega t}\, d(\omega t)\right] \\ &= \frac{V}{2\pi}\left\{-\left[\frac{1}{(-jn)}e^{-jn\omega t}\right]_{-\pi}^{0} + \left[\frac{1}{(-jn)}e^{-jn\omega t}\right]_0^{\pi}\right\} \\ &= \frac{V}{-j2\pi n}(-e^0 + e^{jn\pi} + e^{-jn\pi} - e^0) = j\frac{V}{n\pi}(e^{jn\pi} - 1)\end{aligned}$$

For n even, $e^{jn\pi} = +1$ and $\mathbf{A}_n = 0$; for n odd, $e^{jn\pi} = -1$ and $\mathbf{A}_n = -j(2V/n\pi)$ (half-wave symmetry). The required Fourier series is

$$f(t) = \cdots + j\frac{2V}{3\pi}e^{-j3\omega t} + j\frac{2V}{\pi}e^{-j\omega t} - j\frac{2V}{\pi}e^{j\omega t} - j\frac{2V}{3\pi}e^{j3\omega t} - \cdots$$

The graph in Fig. 17-32 shows amplitudes for both positive and negative frequencies. Combining the values at $+n$ and $-n$ yields the same line spectrum as plotted in Fig. 17-19.

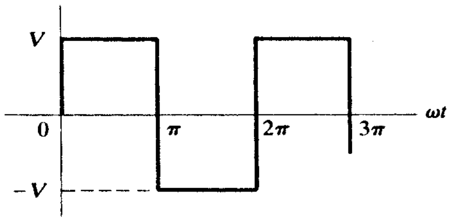


Fig. 17-31

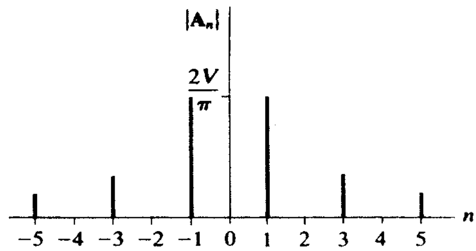


Fig. 71-32

The trigonometric-series cosine coefficients are

$$a_n = 2\,\mathrm{Re}\,\mathbf{A}_n = 0$$

and

$$b_n = -2\,\mathrm{Im}\,\mathbf{A}_n = \frac{4V}{n\pi} \qquad \text{for odd } n \text{ only}$$

These agree with the coefficients obtained in Problem 17.1.

17.9 Find the exponential Fourier series for the triangular wave shown in Figs. 17-20 and 17-33 and sketch the line spectrum.

In the interval $-\pi < \omega t < 0, f(t) = V + (V/\pi)\omega t$; and for $0 < \omega t < \pi, f(t) = V - (V/\pi)\omega t$. The wave is even and therefore the $\mathbf{A}_n$ coefficients will be pure real. By inspection the average value is $V/2$.

$$\begin{aligned}\mathbf{A}_n &= \frac{1}{2\pi}\left\{\int_{-\pi}^{0}[V + (V/\pi)\omega t]e^{-jn\omega t}\,d(\omega t) + \int_0^{\pi}[V - (V/\pi)\omega t]e^{-jn\omega t}\,d(\omega t)\right\} \\ &= \frac{V}{2\pi^2}\left[\int_{-\pi}^{0}\omega t e^{-jn\omega t}\,d(\omega t) + \int_0^{\pi}(-\omega t)e^{-jn\omega t}\,d(\omega t) + \int_{-\pi}^{\pi}\pi e^{-jn\omega t}\,d(\omega t)\right] \\ &= \frac{V}{2\pi^2}\left\{\left[\frac{e^{-jn\omega t}}{(-jn)^2}(-jn\omega t - 1)\right]_{-\pi}^{0} - \left[\frac{e^{-jn\omega t}}{(-jn)^2}(-jn\omega t - 1)\right]_0^{\pi}\right\} = \frac{V}{\pi^2 n^2}(1 - e^{jn\pi})\end{aligned}$$

For even n, $e^{jn\pi} = +1$ and $\mathbf{A}_n = 0$; for odd n, $\mathbf{A}_n = 2V/\pi^2 n^2$. Thus the series is

$$f(t) = \cdots + \frac{2V}{(-3\pi)^2}e^{-j3\omega t} + \frac{2V}{(-\pi)^2}e^{-j\omega t} + \frac{V}{2} + \frac{2V}{(\pi)^2}e^{j\omega t} + \frac{2V}{(3\pi)^2}e^{j3\omega t} + \cdots$$

The harmonic amplitudes

$$c_0 = \frac{V}{2} \qquad c_n = 2|\mathbf{A}_n| = \begin{cases} 0 & (n = 2, 4, 6, \ldots) \\ 4V/\pi^2 n^2 & (n = 1, 3, 5, \ldots)\end{cases}$$

are exactly as plotted in Fig. 17-21.

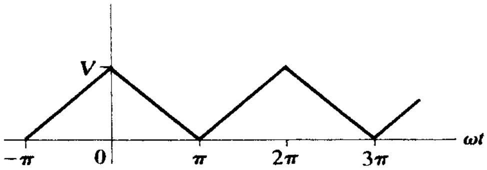


Fig. 17-33

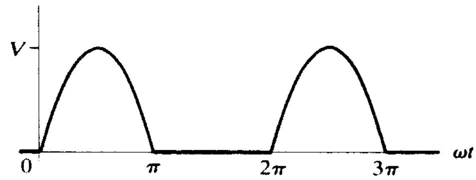


Fig. 17-34

17.10 Find the exponential Fourier series for the half-wave rectified sine wave shown in Figs. 17-26 and 17-34, and sketch the line spectrum.

In the interval $0 < \omega t < \pi$, $f(t) = V \sin \omega t$; and from π to 2π, $f(t) = 0$. Then

$$\mathbf{A}_n = \frac{1}{2\pi}\int_0^{\pi} V \sin \omega t\, e^{-jn\omega t}\, d(\omega t)$$
$$= \frac{V}{2\pi}\left[\frac{e^{-jn\omega t}}{(1-n^2)}(-jn \sin \omega t - \cos \omega t)\right]_0^{\pi} = \frac{V(e^{-jn\pi}+1)}{2\pi(1-n^2)}$$

For even n, $\mathbf{A}_n = V/\pi(1-n^2)$; for odd n, $\mathbf{A}_n = 0$. However, for $n = 1$, the expression for $\mathbf{A}_n$ becomes indeterminate. L'Hôpital's rule may be applied; in other words, the numerator and denominator are separately differentiated with respect to n, after which n is allowed to approach 1, with the result that $\mathbf{A}_1 = -j(V/4)$.

The average value is

$$A_0 = \frac{1}{2\pi}\int_0^{\pi} V \sin \omega t\, d(\omega t) = \frac{V}{2\pi}\Big[-\cos \omega t\Big]_0^{\pi} = \frac{V}{\pi}$$

Then the exponential Fourier series is

$$f(t) = \cdots - \frac{V}{15\pi}e^{-j4\omega t} - \frac{V}{3\pi}e^{-j2\omega t} + j\frac{V}{4}e^{-j\omega t} + \frac{V}{\pi} - j\frac{V}{4}e^{j\omega t} - \frac{V}{3\pi}e^{j2\omega t} - \frac{V}{15\pi}e^{j4\omega t} - \cdots$$

The harmonic amplitudes,

$$c_0 = A_0 = \frac{V}{\pi} \qquad c_n = 2|\mathbf{A}_n| = \begin{cases} 2V/\pi(n^2-1) & (n = 2, 4, 6, \ldots) \\ V/2 & (n = 1) \\ 0 & (n = 3, 5, 7, \ldots) \end{cases}$$

are exactly as plotted in Fig. 17-27.

17.11 Find the average power in a resistance $R = 10\ \Omega$, if the current in Fourier series form is $i = 10 \sin \omega t + 5 \sin 3\omega t + 2 \sin 5\omega t$ (A).

The current has an effective value $I_{\text{eff}} = \sqrt{\frac{1}{2}(10)^2 + \frac{1}{2}(5)^2 + \frac{1}{2}(2)^2} = \sqrt{64.5} = 8.03$ A. Then the average power is $P = I_{\text{eff}}^2 R = (64.5)10 = 645$ W.

Another Method

The total power is the sum of the harmonic powers, which are given by $\frac{1}{2} V_{\max} I_{\max} \cos\theta$. But the voltage across the resistor and the current are in phase for all harmonics, and $\theta_n = 0$. Then,

$$v_R = Ri = 100 \sin \omega t + 50 \sin 3\omega t + 20 \sin 5\omega t$$

and $P = \frac{1}{2}(100)(10) + \frac{1}{2}(50)(5) + \frac{1}{2}(20)(2) = 645$ W.

17.12 Find the average power supplied to a network if the applied voltage and resulting current are

$$v = 50 + 50 \sin 5\times10^3 t + 30 \sin 10^4 t + 20 \sin 2\times10^4 t \quad \text{(V)}$$
$$i = 11.2 \sin(5\times10^3 t + 63.4^\circ) + 10.6 \sin(10^4 t + 45^\circ) + 8.97 \sin(2\times10^4 t + 26.6^\circ) \quad \text{(A)}$$

The total average power is the sum of the harmonic powers:

$$P = (50)(0) + \tfrac{1}{2}(50)(11.2)\cos 63.4^\circ + \tfrac{1}{2}(30)(10.6)\cos 45^\circ + \tfrac{1}{2}(20)(8.97)\cos 26.6^\circ = 317.7 \text{ W}$$

17.13 Obtain the constants of the two-element series circuit with the applied voltage and resultant current given in Problem 17.12.

The voltage series contains a constant term 50, but there is no corresponding term in the current series, thus indicating that one of the elements is a capacitor. Since power is delivered to the circuit, the other element must be a resistor.

$$I_{\text{eff}} = \sqrt{\tfrac{1}{2}(11.2)^2 + \tfrac{1}{2}(10.6)^2 + \tfrac{1}{2}(8.97)^2} = 12.6 \text{ A}$$

The average power is $P = I_{\text{eff}}^2 R$, from which $R = P/I_{\text{eff}}^2 = 317.7/159.2 = 2\ \Omega$.

At $\omega = 10^4$ rad/s, the current leads the voltage by 45°. Hence,

$$1 = \tan 45^\circ = \frac{1}{\omega CR} \qquad \text{or} \qquad C = \frac{1}{(10^4)(2)} = 50\ \mu\text{F}$$

Therefore, the two-element series circuit consists of a resistor of 2 Ω and a capacitor of 50 μF.

17.14 The voltage wave shown in Fig. 17-35 is applied to a series circuit of $R = 2\,\text{k}\Omega$ and $L = 10$ H. Use the trigonometric Fourier series to obtain the voltage across the resistor. Plot the line spectra of the applied voltage and v_R to show the effect of the inductance on the harmonics. $\omega = 377$ rad/s.

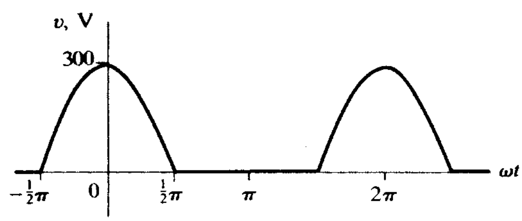


Fig. 17-35

The applied voltage has average value V_{max}/π, as in Problem 17.5. The wave function is even and hence the series contains only cosine terms, with coefficients obtained by the following evaluation integral:

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 300 \cos\omega t \cos n\omega t \, d(\omega t) = \frac{600}{\pi(1-n^2)} \cos n\pi/2 \quad \text{V}$$

Here, $\cos n\pi/2$ has the value -1 for $n = 2, 6, 10, \ldots$, and $+1$ for $n = 4, 8, 12, \ldots$. For n odd, $\cos n\pi/2 = 0$. However, for $n = 1$, the expression is indeterminate and must be evaluated separately.

$$a_1 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 300\cos^2 \omega t \, d(\omega t) = \frac{300}{\pi}\left[\frac{\omega t}{2} + \frac{\sin 2\omega t}{4}\right]_{-\pi/2}^{\pi/2} = \frac{300}{2} \text{ V}$$

Thus,
$$v = \frac{300}{\pi}\left(1 + \frac{\pi}{2}\cos\omega t + \frac{2}{3}\cos 2\omega t - \frac{2}{15}\cos 4\omega t + \frac{2}{35}\cos 6\omega t - \cdots\right) \quad \text{(V)}$$

In Table 17-3, the total impedance of the series circuit is computed for each harmonic in the voltage expression. The Fourier coefficients of the current series are the voltage series coefficients divided by the Z_n; the current terms lag the voltage terms by the phase angles θ_n.

Table 17-3

n	$n\omega$, rad/s	R, kΩ	$n\omega L$, kΩ	Z_n, kΩ	θ_n
0	0	2	0	2	0°
1	377	2	3.77	4.26	62°
2	754	2	7.54	7.78	75.1°
4	1508	2	15.08	15.2	82.45°
6	2262	2	22.62	22.6	84.92°

$$I_0 = \frac{300/\pi}{2} \text{ mA}$$

$$i_1 = \frac{300/2}{4.26} \cos(\omega t - 62°) \quad \text{(mA)}$$

$$i_2 = \frac{600/3\pi}{7.78} \cos(2\omega t - 75.1°) \quad \text{(mA)}$$

....

Then the current series is

$$i = \frac{300}{2\pi} + \frac{300}{(2)(4.26)} \cos(\omega t - 62°) + \frac{600}{3\pi(7.78)} \cos(2\omega t - 75.1°)$$
$$- \frac{600}{15\pi(15.2)} \cos(4\omega t - 82.45°) + \frac{600}{35\pi(22.6)} \cos(6\omega t - 84.92°) - \cdots \quad \text{(mA)}$$

and the voltage across the resistor is

$$v_R = Ri = 95.5 + 70.4 \cos(\omega t - 62°) + 16.4 \cos(2\omega t - 75.1°)$$
$$- 1.67 \cos(4\omega t - 82.45°) + 0.483 \cos(6\omega t - 84.92°) - \cdots \quad \text{(V)}$$

Figure 17-36 shows clearly how the harmonic amplitudes of the applied voltage have been reduced by the 10-H series inductance.

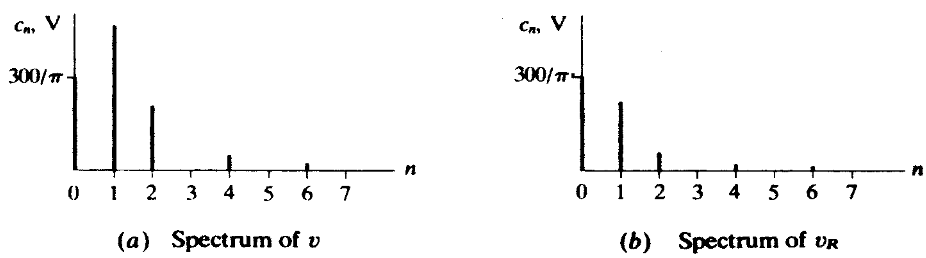


Fig. 17-36

17.15 The current in a 10-mH inductance has the waveform shown in Fig. 17-37. Obtain the trigonometric series for the voltage across the inductance, given that $\omega = 500$ rad/s.

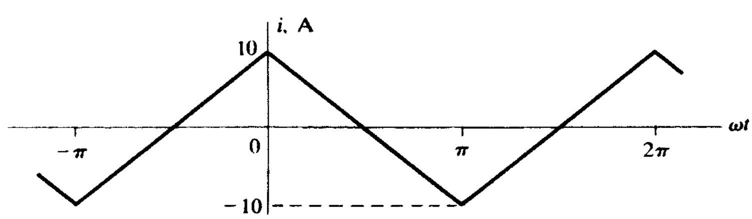


Fig. 17-37

The derivative of the waveform of Fig. 17-37 is graphed in Fig. 17-38. This is just Fig. 17-18 with $V = -20/\pi$. Hence, from Problem 17.1,

$$\frac{di}{d(\omega t)} = -\frac{80}{\pi^2}\left(\sin\omega t + \tfrac{1}{3}\sin 3\omega t + \tfrac{1}{5}\sin 5\omega t + \cdots\right) \quad \text{(A)}$$

and so

$$v_L = L\omega\frac{di}{d(\omega t)} = -\frac{400}{\pi^2}\left(\sin\omega t + \tfrac{1}{3}\sin 3\omega t + \tfrac{1}{5}\sin 5\omega t + \cdots\right) \quad \text{(V)}$$

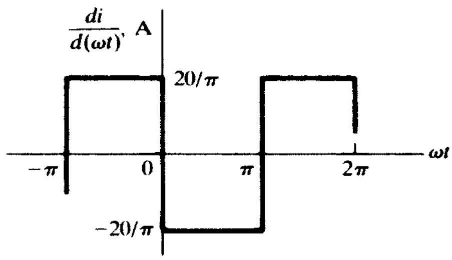


Fig. 17-38

Supplementary Problems

17.16 Synthesize the waveform for which the trigonometric Fourier series is

$$f(t) = \frac{8V}{\pi^2}\{\sin\omega t - \tfrac{1}{9}\sin 3\omega t + \tfrac{1}{25}\sin 5\omega t - \tfrac{1}{49}\sin 7\omega t + \cdots\}$$

17.17 Synthesize the waveform if its Fourier series is

$$f(t) = 5 - \frac{40}{\pi^2}\left(\cos\omega t + \tfrac{1}{9}\cos 3\omega t + \tfrac{1}{25}\cos 5\omega t + \cdots\right) + \frac{20}{\pi}\left(\sin\omega t - \tfrac{1}{2}\sin 2\omega t + \tfrac{1}{3}\sin 3\omega t - \tfrac{1}{4}\sin 4\omega t + \cdots\right)$$

17.18 Synthesize the waveform for the given Fourier series.

$$f(t) = V\left(\frac{1}{2\pi} - \frac{1}{\pi}\cos\omega t - \frac{1}{3\pi}\cos 2\omega t + \frac{1}{2\pi}\cos 3\omega t - \frac{1}{15\pi}\cos 4\omega t - \frac{1}{6\pi}\cos 6\omega t + \cdots + \frac{1}{4}\sin\omega t - \frac{2}{3\pi}\sin 2\omega t + \frac{4}{15\pi}\sin 4\omega t - \cdots\right)$$

17.19 Find the trigonometric Fourier series for the sawtooth wave shown in Fig. 17-39 and plot the line spectrum. Compare with Example 17.1.

Ans. $f(t) = \dfrac{V}{2} + \dfrac{V}{\pi}\left(\sin\omega t + \tfrac{1}{2}\sin 2\omega t + \tfrac{1}{3}\sin 3\omega t + \cdots\right)$

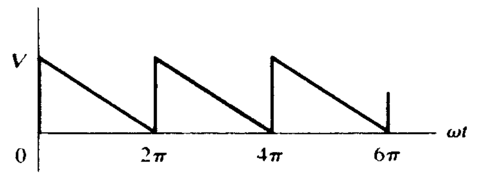


Fig. 17-39

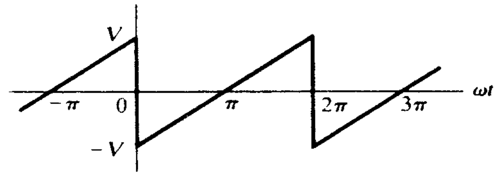


Fig. 17-40

17.20 Find the trigonometric Fourier series for the sawtooth wave shown in Fig. 17-40 and plot the spectrum. Compare with the result of Problem 17.3.

Ans. $f(t) = \dfrac{-2V}{\pi}\{\sin \omega t + \tfrac{1}{2}\sin 2\omega t + \tfrac{1}{3}\sin 3\omega t + \tfrac{1}{4}\sin 4\omega t + \cdots\}$

17.21 Find the trigonometric Fourier series for the waveform shown in Fig. 17-41 and plot the line spectrum.

Ans. $f(t) = \dfrac{4V}{\pi^2}\{\cos \omega t + \tfrac{1}{9}\cos 3\omega t + \tfrac{1}{25}\cos 5\omega t + \cdots\} - \dfrac{2V}{\pi}\{\sin \omega t + \tfrac{1}{3}\sin 3\omega t + \tfrac{1}{5}\sin 5\omega t + \cdots\}$

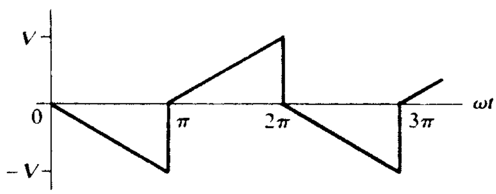


Fig. 17-41

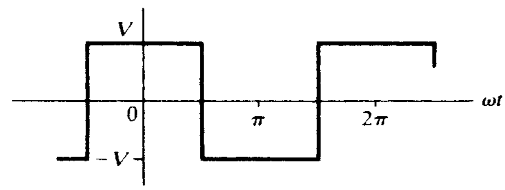


Fig. 17-42

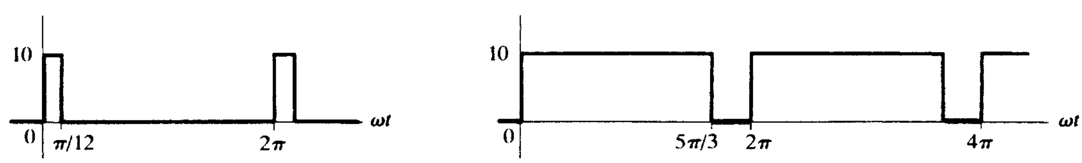
17.22 Find the trigonometric Fourier series of the square wave shown in Fig. 17-42 and plot the line spectrum. Compare with the result of Problem 17.1.

Ans. $f(t) = \dfrac{4V}{\pi}\{\cos \omega t - \tfrac{1}{3}\cos 3\omega t + \tfrac{1}{5}\cos 5\omega t - \tfrac{1}{7}\cos 7\omega t + \cdots\}$

17.23 Find the trigonometric Fourier series for the waveforms shown in Fig. 17-43. Plot the line spectrum of each and compare.

Ans. (*a*) $f(t) = \dfrac{5}{12} + \displaystyle\sum_{n=1}^{\infty}\left[\frac{10}{n\pi}\left(\sin\frac{n\pi}{12}\right)\cos n\omega t + \frac{10}{n\pi}\left(1 - \cos\frac{n\pi}{12}\right)\sin n\omega t\right]$

(*b*) $f(t) = \dfrac{50}{6} + \displaystyle\sum_{n=1}^{\infty}\left[\frac{10}{n\pi}\left(\sin\frac{n5\pi}{3}\right)\cos n\omega t + \frac{10}{n\pi}\left(1 - \cos\frac{n5\pi}{3}\right)\sin n\omega t\right]$



(*a*) (*b*)

Fig. 17-43

17.24 Find the trigonometric Fourier series for the half-wave-rectified sine wave shown in Fig. 17-44 and plot the line spectrum. Compare the answer with the results of Problems 17.5 and 17.6.

Ans. $f(t) = \dfrac{V}{\pi}\left(1 + \dfrac{\pi}{2}\cos \omega t + \dfrac{2}{3}\cos 2\omega t - \dfrac{2}{15}\cos 4\omega t + \dfrac{2}{35}\cos 6\omega t - \cdots\right)$

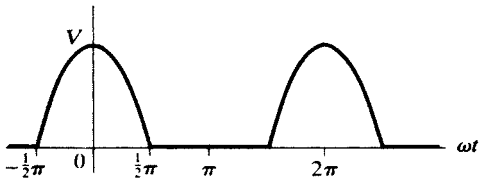


Fig. 17-44

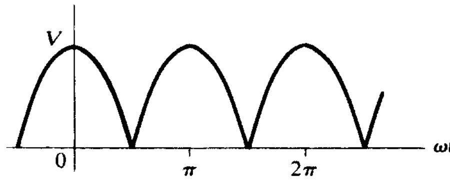


Fig. 17-45

17.25 Find the trigonometric Fourier series for the full-wave-rectified sine wave shown in Fig. 17-45 and plot the spectrum.

Ans. $f(t) = \dfrac{2V}{\pi}\,(1 + \frac{2}{3}\cos 2\omega t - \frac{2}{15}\cos 4\omega t + \frac{2}{35}\cos 6\omega t - \cdots)$

17.26 The waveform in Fig. 17-46 is that of Fig. 17-45 with the origin shifted. Find the Fourier series and show that the two spectra are identical.

Ans. $f(t) = \dfrac{2V}{\pi}\,(1 - \frac{2}{3}\cos 2\omega t - \frac{2}{15}\cos 4\omega t - \frac{2}{35}\cos 6\omega t - \cdots)$

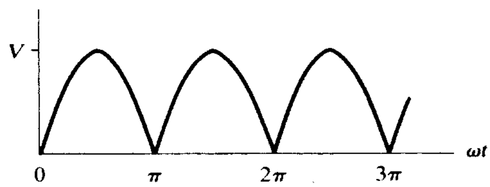


Fig. 17-46

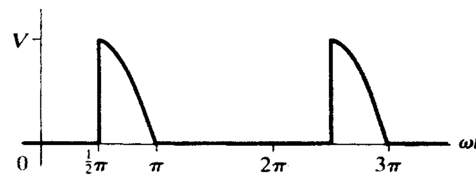


Fig. 17-47

17.27 Find the trigonometric Fourier series for the waveform shown in Fig. 17-47.

Ans.
$$f(t) = \frac{V}{2\pi} - \frac{V}{2\pi}\cos\omega t + \sum_{n=2}^{\infty} \frac{V}{\pi(1-n^2)}\,(\cos n\pi + n \sin n\pi/2)\cos n\omega t$$
$$+ \frac{V}{4}\sin\omega t + \sum_{n=2}^{\infty}\left[\frac{-nV\cos n\pi/2}{\pi(1-n^2)}\right]\sin n\omega t$$

17.28 Find the trigonometric Fourier series for the waveform shown in Fig. 17-48. Add this series termwise to that of Problem 17.27, and compare the sum with the series obtained in Problem 17.5.

Ans.
$$f(t) = \frac{V}{2\pi} + \frac{V}{2\pi}\cos\omega t + \sum_{n=2}^{\infty}\frac{V(n\sin n\pi/2 - 1)}{\pi(n^2-1)}\cos n\omega t + \frac{V}{4}\sin\omega t + \sum_{n=2}^{\infty}\frac{nV\cos n\pi/2}{\pi(1-n^2)}\sin n\omega t$$

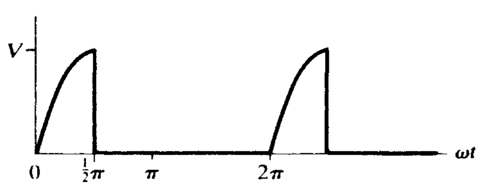


Fig. 17-48

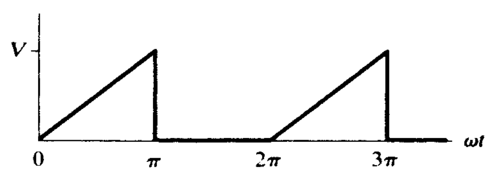


Fig. 17-49

17.29 Find the exponential Fourier series for the waveform shown in Fig. 17-49 and plot the line spectrum. Convert the coefficients obtained here into the trigonometric series coefficients, write the trigonometric series, and compare it with the result of Problem 17.4.

Ans. $$f(t) = V\left[\cdots + \left(\frac{1}{9\pi^2} - j\frac{1}{6\pi}\right)e^{-j3\omega t} - j\frac{1}{4\pi}e^{-j2\omega t} - \left(\frac{1}{\pi^2} - j\frac{1}{2\pi}\right)e^{-j\omega t} + \frac{1}{4} - \left(\frac{1}{\pi^2} + j\frac{1}{2\pi}\right)e^{j\omega t} + j\frac{1}{4\pi}e^{j2\omega t} - \left(\frac{1}{9\pi^2} + j\frac{1}{6\pi}\right)e^{j3\omega t} - \cdots\right]$$

17.30 Find the exponential Fourier series for the waveform shown in Fig. 17-50 and plot the line spectrum.

Ans. $$f(t) = V\left[\cdots + \left(\frac{1}{9\pi^2} + j\frac{1}{6\pi}\right)e^{-j3\omega t} + j\frac{1}{4\pi}e^{-j2\omega t} + \left(\frac{1}{\pi^2} + j\frac{1}{2\pi}\right)e^{-j\omega t} + \frac{1}{4} + \left(\frac{1}{\pi^2} - j\frac{1}{2\pi}\right)e^{j\omega t} - j\frac{1}{4\pi}e^{j2\omega t} + \left(\frac{1}{9\pi^2} - j\frac{1}{6\pi}\right)e^{j3\omega t} + \cdots\right]$$

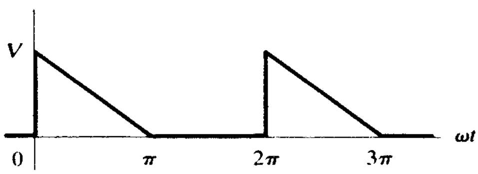


Fig. 17-50

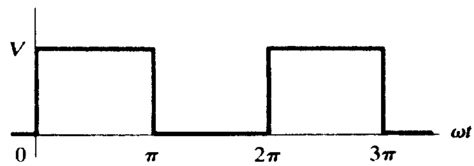


Fig. 17-51

17.31 Find the exponential Fourier series for the square wave shown in Fig. 17-51 and plot the line spectrum. Add the exponential series of Problems 17.29 and 17.30 and compare the sum to the series obtained here.

Ans. $$f(t) = V\left(\cdots + j\frac{1}{3\pi}e^{-j3\omega t} + j\frac{1}{\pi}e^{-j\omega t} + \frac{1}{2} - j\frac{1}{\pi}e^{j\omega t} - j\frac{1}{3\pi}e^{j3\omega t} - \cdots\right)$$

17.32 Find the exponential Fourier series for the sawtooth waveform shown in Fig. 17-52 and plot the spectrum. Convert the coefficients obtained here into the trigonometric series coefficients, write the trigonometric series, and compare the results with the series obtained in Problem 17.19.

Ans. $$f(t) = V\left(\cdots + j\frac{1}{4\pi}e^{-j2\omega t} + j\frac{1}{2\pi}e^{-j\omega t} + \frac{1}{2} - j\frac{1}{2\pi}e^{j\omega t} - j\frac{1}{4\pi}e^{j2\omega t} - \cdots\right)$$

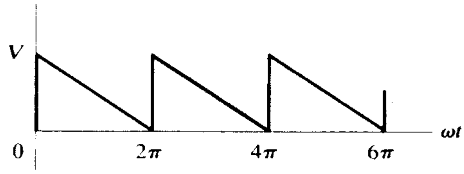


Fig. 17-52

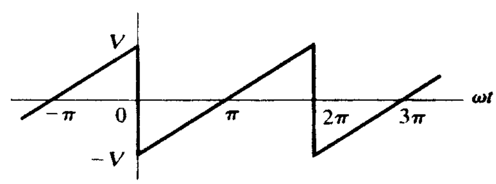


Fig. 17-53

17.33 Find the exponential Fourier series for the waveform shown in Fig. 17-53 and plot the spectrum. Convert the trigonometric series coefficients found in Problem 17.20 into exponential series coefficients and compare them with the coefficients of the series obtained here.

Ans. $$f(t) = V\left(\cdots - j\frac{1}{2\pi}e^{-j2\omega t} - j\frac{1}{\pi}e^{-j\omega t} + j\frac{1}{\pi}e^{j\omega t} + j\frac{1}{2\pi}e^{j2\omega t} + \cdots\right)$$

17.34 Find the exponential Fourier series for the waveform shown in Fig. 17-54 and plot the spectrum. Convert the coefficients to trigonometric series coefficients, write the trigonometric series, and compare it with that obtained in Problem 17.21.

Ans. $$f(t) = V\left[\cdots + \left(\frac{2}{9\pi^2} - j\frac{1}{3\pi}\right)e^{-j3\omega t} + \left(\frac{2}{\pi^2} - j\frac{1}{\pi}\right)e^{-j\omega t} + \left(\frac{2}{\pi^2} + j\frac{1}{\pi}\right)e^{j\omega t} + \left(\frac{2}{9\pi^2} + j\frac{1}{3\pi}\right)e^{j3\omega t} + \cdots\right]$$

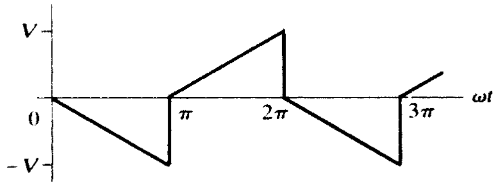


Fig. 17-54

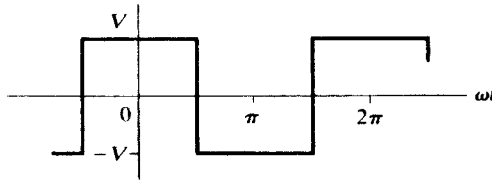


Fig. 17-55

17.35 Find the exponential Fourier series for the square wave shown in Fig. 17-55 and plot the line spectrum. Convert the trigonometric series coefficients of Problem 17.22 into exponential series coefficients and compare with the coefficients in the result obtained here.

Ans. $$f(t) = \frac{2V}{\pi}\left(\cdots + \tfrac{1}{5}e^{-j5\omega t} - \tfrac{1}{3}e^{-j3\omega t} + e^{-j\omega t} + e^{j\omega t} - \tfrac{1}{3}e^{-j3\omega t} + \tfrac{1}{5}e^{j5\omega t} - \cdots\right)$$

17.36 Find the exponential Fourier series for the waveform shown in Fig. 17-56 and plot the line spectrum.

Ans. $$f(t) = \cdots + \frac{V}{2\pi}\sin\left(\frac{2\pi}{6}\right)e^{-j2\omega t} + \frac{V}{\pi}\sin\left(\frac{\pi}{6}\right)e^{-j\omega t} + \frac{V}{6} + \frac{V}{\pi}\sin\left(\frac{\pi}{6}\right)e^{j\omega t} + \frac{V}{2\pi}\sin\left(\frac{2\pi}{6}\right)e^{j2\omega t} + \cdots$$

Fig. 17-56

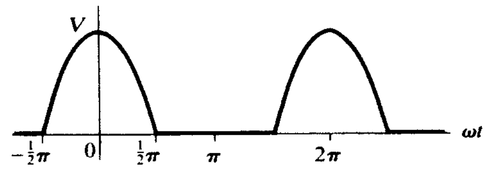


Fig. 17-57

17.37 Find the exponential Fourier series for the half-wave-rectified sine wave shown in Fig. 17-57. Convert these coefficients into the trigonometric series coefficients, write the trigonometric series, and compare it with the result of Problem 17.24.

Ans. $$f(t) = \cdots - \frac{V}{15\pi}e^{-j4\omega t} + \frac{V}{3\pi}e^{-j2\omega t} + \frac{V}{4}e^{-j\omega t} + \frac{V}{\pi} + \frac{V}{4}e^{j\omega t} + \frac{V}{3\pi}e^{j2\omega t} - \frac{V}{15\pi}e^{j4\omega t} + \cdots$$

17.38 Find the exponential Fourier series for the full-wave rectified sine wave shown in Fig. 17-58 and plot the line spectrum.

Ans. $$f(t) = \cdots - \frac{2V}{15\pi}e^{-j4\omega t} + \frac{2V}{3\pi}e^{-j2\omega t} + \frac{2V}{\pi} + \frac{2V}{3\pi}e^{j2\omega t} - \frac{2V}{15\pi}e^{j4\omega t} + \cdots$$

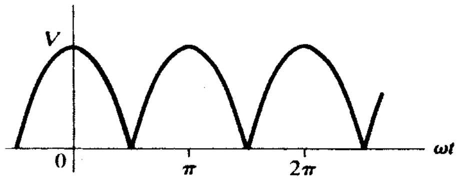


Fig. 17-58

17.39 Find the effective voltage, effective current, and average power supplied to a passive network if the applied voltage is $v = 200 + 100\cos(500t + 30°) + 75\cos(1500t + 60°)$ (V) and the resulting current is $i = 3.53\cos(500t + 75°) + 3.55\cos(1500t + 78.45°)$ (A). *Ans.* 218.5 V, 3.54 A, 250.8 W

17.40 A voltage $v = 50 + 25\sin 500t + 10\sin 1500t + 5\sin 2500t$ (V) is applied to the terminals of a passive network and the resulting current is

$$i = 5 + 2.23\sin(500t - 26.6°) + 0.556\sin(1500t - 56.3°) + 0.186\sin(2500t - 68.2°) \quad \text{(A)}$$

Find the effective voltage, effective current, and the average power. *Ans.* 53.6 V, 5.25 A, 276.5 W

17.41 A three-element series circuit, with $R = 5\ \Omega$, $L = 5$ mH, and $C = 50\ \mu$F, has an applied voltage $v = 150\sin 1000t + 100\sin 2000t + 75\sin 3000t$ (V). Find the effective current and the average power for the circuit. Sketch the line spectrum of the voltage and the current, and note the effect of series resonance. *Ans.* 16.58 A, 1374 W

17.42 A two-element series circuit, with $R = 10\ \Omega$ and $L = 20$ mH, has current

$$i = 5\sin 100t + 3\sin 300t + 2\sin 500t \quad \text{(A)}$$

Find the effective applied voltage and the average power. *Ans.* 48 V, 190 W

17.43 A pure inductance, $L = 10$ mH, has the triangular current wave shown in Fig. 17-59, where $\omega = 500$ rad/s. Obtain the exponential Fourier series for the voltage across the inductance. Compare the answer with the result of Problem 17.8.

Ans. $v_L = \dfrac{200}{\pi^2}(\cdots - j\frac{1}{3}e^{-j3\omega t} - je^{-j\omega t} + je^{j\omega t} + j\frac{1}{3}e^{j\omega t} + \cdots)$ (V)

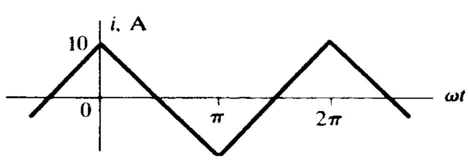


Fig. 17-59

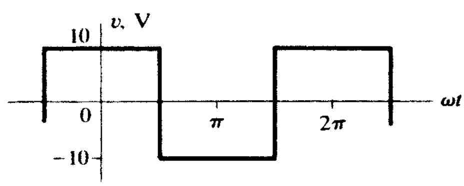


Fig. 17-60

17.44 A pure inductance, $L = 10$ mH, has an applied voltage with the waveform shown in Fig. 17-60, where $\omega = 200$ rad/s. Obtain the current series in trigonometric form and identify the current waveform.

Ans. $i = \dfrac{20}{\pi}(\sin\omega t - \frac{1}{9}\sin 3\omega t + \frac{1}{25}\sin 5\omega t - \frac{1}{49}\sin 7\omega t + \cdots)$ (A); triangular

17.45 Figure 17-61 shows a full-wave-rectified sine wave representing the voltage applied to the terminals of an *LC* series circuit. Use the trigonometric Fourier series to find the voltages across the inductor and the capacitor.

Ans. $$v_L = \frac{4V_m}{\pi}\left[\frac{2\omega L}{3\left(2\omega L - \dfrac{1}{2\omega C}\right)}\cos 2\omega t - \frac{4\omega L}{15\left(4\omega L - \dfrac{1}{4\omega C}\right)}\cos 4\omega t + \cdots\right]$$

$$v_C = \frac{4V_m}{\pi}\left[\frac{1}{2} - \frac{1}{3(2\omega C)\left(2\omega L - \dfrac{1}{2\omega C}\right)}\cos 2\omega t + \frac{1}{15(4\omega C)\left(4\omega L - \dfrac{1}{4\omega C}\right)}\cos 4\omega t - \cdots\right]$$

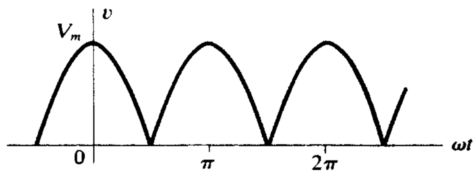


Fig. 17-61

17.46 A three-element circuit consists of $R = 5\ \Omega$ in series with a parallel combination of L and C. At $\omega = 500$ rad/s, $X_L = 2\ \Omega$, $X_C = 8\ \Omega$. Find the total current if the applied voltage is given by $v = 50 + 20\sin 500t + 10\sin 1000t$ (V). *Ans.* $i = 10 + 3.53\sin(500t - 28.1^\circ)$ (A)