Assignment 4

Due March 19, 2015 at the beginning of lecture. Marked out of 30 and worth 10% of your final mark.

1. **[0 marks]:** Read Chapters 17, 18. Do problems: 17.3, 17.4, 17.5, 18.2, 18.3, 18.5, 18.7.

2. **[20 marks]:** Consider an isothermal, ideal gas sphere, embedded within a low-density, hot medium. As with our study of stellar structure, we are interested in equilibrium configurations, but in this case we can no longer assume that the surface pressure vanishes. In the classic paper W.B. Bonnor, MNRAS, 116, 351 (1956), Bonnor showed that pressure-confined isothermal gas spheres would become unstable if sufficiently compressed. This is intimately related to the Jeans' instability, and here we will reproduce some of the relevant calculations. These represent models of globules, the sites of star formation, within molecular clouds.

   a) (5 marks) In principle, an equilibrium isothermal, ideal gas sphere is fully characterized by two parameters: a mass and temperature. However, these are even more simple in practice, forming a one-dimensional family of structures that then may be rescaled as needed. This may be illustrated by conveniently "rationalizing" the equations of hydrostatic equilibrium. That is, defining, dimensionless versions of the enclosed mass, radius, and density:

   $$\mathcal{M} = \tilde{\mathcal{M}} M_0 \,, \quad r = \tilde{r}\frac{GM_0\mu m_p}{kT} \,, \quad \text{and} \quad \rho = \tilde{\rho} M_0 \left(\frac{kT}{GM_0\mu m_p}\right)^3 \,, \tag{1}$$

   respectively, where M_0 is a mass scale (not necessarily the total mass!) and T is the temperature. Using the equations of hydrostatic equilibrium, obtain differential equations for $\tilde{\rho}$ and $\tilde{M}$ in terms of $\tilde{\rho}$, $\tilde{M}$, and $\tilde{r}$.

   b) (5 marks) Solve the equations in part (a) numerically, choosing $\tilde{\rho}(0) = 1$, to obtain $\tilde{\rho}(\tilde{r})$ and $\tilde{\mathcal{M}}(\tilde{r})$. Plot these for $\tilde{r} \in (0, 5]$. *Note that this is simply a special case of the set of solutions you found in the previous assignment, with $\gamma = 1$ or, equivalently, $n = \infty$. However, since n diverges, solving these via the Lane-Embden equation is not possible. Nevertheless, you will be able to solve them in a very similar way with the same sort of numerical alogrithm.*

   c) (5 marks) Up to some rescalings, your solution in part b provides the structure of the entire family of isothermal gas spheres; spheres in different ambient media will be truncated at different radii! For example, $\mathcal{M}(r)$ represents the mass of an isothermal gas sphere of radius r, truncated at a surface pressure of $P_s = \rho(r)kT/\mu m_p$. However, in our rationalized expressions we can also renormalize among different total masses, total radii, and temperatures! Hence, for any combination of cloud temperature (T) and mass (M), we may choose an $\tilde{r}$ and M_0 such that our rationalized solution may be simply rescaled. Explicitly, to generate a family of fixed-mass spheres with mass M and fixed temperature T, we set

   $$M_0 \tilde{\mathcal{M}}(\tilde{r}) = M \quad \rightarrow \quad M_0 = M/\tilde{\mathcal{M}}(\tilde{r}) \,, \tag{2}$$

   from which we recover the unscaled radius,

   $$r = \tilde{r}\frac{GM\mu m_p}{kT\tilde{\mathcal{M}}(\tilde{r})} \,. \tag{3}$$

   Using this, construct the surface pressure of a sequence of gas spheres with fixed mass $M = 1M_\odot$ and temperature $T = 10$ K. You may assume that the mean molecular weight is 2.4. Plot the surface pressure as a function of radius (in AU). *Hint: This family of solutions may be parameterized in a number of different ways, however, in practice it will be easiest if we choose to do so in terms of $\tilde{r}$. See Fig. I of Bonnor (1956) to get an idea if what you have makes sense.*

d) (5 marks) Now consider simply a $1 M_\odot$ gas sphere, enclosed within a radius R. This is a gross mock-up of the isothermal sphere computation we performed above. Compute and plot the Jeans' mass associated with the average gas sphere density (i.e., $\bar{\rho} \equiv 3M/(4\pi R^3)$) as a function of sphere radius. What happens when the sphere radius is compressed below the pressure peak near $r = 10^4$ AU?

3. **[10 marks]:** In the computation of the Jeans' mass we considered only thermal pressure as a means to stop gravitational collapse. However, in many molecular clouds, magnetic pressure is believed to play a critical, if not dominant, role in supporting against collapse. Here we consider the ability of magnetic stresses to do this.

a) (2 marks) Recall that the Jeans' mass, M_J, was obtained by relating the sound crossing time to the free-fall time:

$$\frac{c_s}{R_J} = \frac{1}{t_{\rm ff}} \quad \Rightarrow \quad R_J = c_s t_{\rm ff} \quad \Rightarrow \quad M_J \equiv \frac{4\pi}{3} R_J^3 \, \rho \propto \frac{c_s^3}{\rho^{1/2}} \,, \tag{4}$$

since $t_{\rm ff} \propto \rho^{-1/2}$. If we consider an initially unstable, uniform density cloud of gas undergoing an isothermal collapse, this implies that where thermal pressure dominates, M_J is related to the cloud radius, R, via a power law, i.e., $M_J \propto R^\eta$. What is η? What does this imply for the collapse dynamics (i.e., is the cloud becoming more or less unstable as R shrinks?)? You may assume that the cloud remains uniform during collapse.

b) (1 marks) The critical physical input into the Jeans' mass estimate was that changes in the thermal pressure can only be propagated at speeds less than c_s. If magnetic fields are present, magnetic stresses can be propagated at the Alfvén velocity, $v_A \equiv \sqrt{B^2/\mu_0 \rho}$, where μ_0 is the permeability of free space. If B evolves solely due to flux conservation, i.e., $B \propto \Phi/R^2 \propto R^{-2}$, where Φ is the magnetic flux through some surface in the cloud, how does v_A/c_s vary with R? *Hint: again this should be a power-law.*

c) (1 marks) At some point, the magnetic stresses will begin to dominate the thermal pressures. This occurs when the magnetic pressure, $P_B = B^2/2\mu_0$, becomes larger than the thermal pressure $P_T = \rho kT/\mu m_p \simeq \rho c_s^2/2$. In terms of the initial cloud size and the initial ratio of the Alfvén velocity to the sound speed, at what radius, R_A, does this occur? For comparison, typically, v_A/c_s in the ISM is somewhere around 0.01–0.1.

d) (3 marks) When $v_A > c_s$, the magnetic stresses can be transmitted more rapidly than thermal pressures, and thus the magnetic fields set the instability scale. Replacing c_s with v_A in the derivation above, what is η now? Sketch M_J as a function of R throughout a collapse, beginning with $R_0 > R_A$ and extending through $R < R_A$. Can a flux-frozen magnetic field stabilize an initially unstable gas cloud alone?

e) (3 mark) When molecular clouds collapse they produce stars, which then inject additional energy and momentum into the clouds from which they form, e.g., via stellar winds and supernovae. This generates turbulence, and subsequently amplifies the magnetic field, and thus our assumption that the magnetic flux remained fixed is violated below the scale at which stars are formed. Making the rather over-simplified assumption that $\Phi \propto R^{-\alpha}$, how does M_J depend upon R now (i.e., what is η now)? Again, sketch M_J, assuming that $\alpha = 2$ and the radius at which rapid star formation begins to occur is $R_{\rm sf} < R_A < R_0$. What does this imply for molecular cloud stability?