Assignment 3

Due March 5, 2015 at the beginning of lecture. Marked out of 30 and worth 10% of your final mark.

1. [**0 marks**]: Read Chapter 15. Do problems: 15.1, 15.3, 15.4, 15.5, 15.10

2. [**5 marks**]: Use dimensional analysis to calculate the exponents α and β in the scaling relation $L \propto M^{\alpha} R^{\beta}$, in the case of:

   a) A purely radiative star, with an opacity given by Kramer's law, $\kappa \propto \rho T^{-3.5}$.

   b) A purely radiative star, with a polytropic equation of state $P \propto \rho^{\gamma}$, and $\gamma = 5/3$. Keep the assumption that the opacity is given by Kramer's law.

   c) Assuming energy production dominated by the PP chain, derive a scaling relation for luminosity as a function of mass only, in each of the two scenarios above.

3. [**25 marks**] Complete the following problem:
   The Lane-Emden equation gives a solution for the structure of a stellar interior.

   a) (2 marks) Show that the equation of hydrostatic equilibrium and the equation of mass continuity can be combined to give:

   $$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho. \tag{1}$$

   b) (1 mark) Now assume a polytropic equation of state, $P = K\rho^{\gamma}$, where K is a constant. Show that Equation 1 becomes

   $$\frac{K\gamma}{4\pi G}\frac{1}{r^2}\frac{d}{dr}\left(r^2\rho^{\gamma-2}\frac{d\rho}{dr}\right) = -\rho. \tag{2}$$

   This is a second-order ordinary differential equation for ρ, and can be solved numerically.

   c) (1 mark) Make the variable substitutions

   $$x = r/\alpha \tag{3}$$

   and

   $$\theta = \left(\frac{\rho}{\rho_c}\right)^{\gamma-1}, \tag{4}$$

   where ρ_c is the central density and α is a collection of constants. Show that, with an appropriate choice of α,

   $$\frac{1}{x^2}\frac{d}{dx}\left[x^2\frac{d\theta}{dx}\right] = -\theta^n \tag{5}$$

   or, in integral form:

   $$x^2\frac{d\theta}{dx} = -\int x^2\theta^n dx, \tag{6}$$

   where $n = 1/(\gamma - 1)$. This is known as the *Lane-Emden equation.*

   d) (2 marks) Determine and justify appropriate boundary conditions for θ and $\dot{\theta}$ at $x = 0$.

   e) (3 marks) Write a program to calculate θ as a function of x, for any input value of n. You will have to do this numerically (analytic solutions are only possible for n=0, n=1 and n=5). See the hints below for help doing this. Plot θ as a function of x for $n = 0, 1, 2, 3, 4$ and 5.

f) (3 marks) Write the mass continuity equation as an integral equation, in terms of the variables α, x, ρ_c and $d\theta/dx$. Combine with the integral form of the Lane-Emden equation to obtain an expression for M_r as a function of these variables (you don't need to integrate anything).

g) (3 marks) **For this, and all following parts, assume n=3.** Assume the outer radius of the star $r = R_{\rm star}$ corresponds to $x_\circ$, where the density (θ) is equal to zero. Use your numerical solution of $\theta(x)$ to find a numerical value of α for $R_{\rm star} = 1R_{\rm Sun}$. Using your result from part f), calculate ρ_c for a solar mass star (i.e. $M(R_{\rm star}) = 1M_{\rm Sun}$). Finally, you can solve for the constant K.

h) (4 marks) Now compute ρ, P and T as a function of $r/R_{\rm star}$ and show the results graphically. Assume a fully ionized gas with mass fractions $X = 0.55$ and $Y = 0.4$ (independent of radius).

i) (4 marks) Calculate the energy generation rates due to the proton-proton chain and the CNO cycle (assume $X_{\rm CNO} = 0.03X$), as a function of $r/R_{\rm star}$. Plot dL/dr as a function of $r/R_{\rm star}$. From this, calculate the total luminosity of the star, at $r = R_{\rm star}$.

j) (2 marks) Use your code to compute the total luminosity, central temperature and central density for i) a star with $M_{\rm star} = 20M_{\rm Sun}$ and $R_{\rm star} = 10R_{\rm Sun}$ and ii) a star with $M_{\rm star} = 0.5M_{\rm Sun}$ and $R_{\rm star} = 0.6R_{\rm Sun}$. Compare these with your results for the solar-mass star.

Hints for the numerical integration:

1. Start with a large vector of x values, ranging from 0 to 7, linearly spaced.
2. Using appropriate boundary conditions for $\theta(0)$ and $\dot{\theta}(0)$, solve for $\ddot{\theta}(0)$.
3. Use your value of $\ddot{\theta}(0)$ to estimate the value of $\dot{\theta}(\Delta x)$. Similarly, use the value of $\dot{\theta}(0)$ to estimate the value of $\theta(\Delta x)$.
4. Compute $\ddot{\theta}(\Delta x)$ from $\dot{\theta}(\Delta x)$ and $\theta(\Delta x)$.
5. Iterate steps 3-4 until you reach $\theta(x_\circ) = 0$.